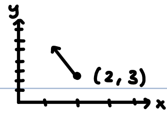
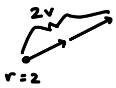
[2/3/25 - recitation 1]

VECTORS: in real n space $(\mathbb{R}^n)$ a point is described by n real coordinates

→ an arrow can also be described by n real coordinates

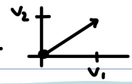


OPERATIONS on vectors:

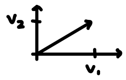
	geometric pic	in coords.
→ SCALAR MULTIPLICATION: given a real # r & vector v,		$r \cdot (v_1, \ldots, v_n) = (rv_1, \ldots, rv_n)$
→ ADDITION: given v & w		$(v_1, \ldots, v_n) + (w_1, \ldots, w_n) = (v_1 + w_1, \ldots, v_n + w_n)$

→ SUBTRACTION:
$v - w = v + (-w)$

→ LENGTH: what is length of $v = (v_1, v_2)$? $\sqrt{v_1^2 + v_2^2}$



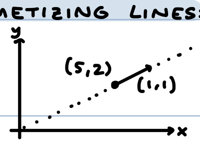
in general, length of $v = (v_1, \ldots, v_n)$ is $\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$



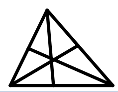
→ UNIT VECTORS: vector of length 1

given a vector $\vec{v}$ in direction of $\vec{v}$ is $\hat{v} = \frac{\vec{v}}{|v|}$

→ PARAMETRIZING LINES:



line is $(5, 2) + t(1, 1)$

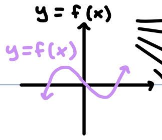


[2/4/25 - lecture 1] → [· single vs. multivariable
· vectors, dot prod., plane eqns

SINGLE VS. MULTIVARIABLE CALC:

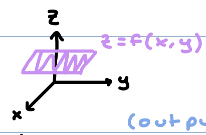
single:

$y = f(x)$



multi:

→ $z = f(x, y)$

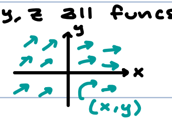


4D-~~graphs~~

→ $w = f(x, y, z)$

→ parameterized curves: x, y, z all funcs. of parameter t → 1 input, 3 outputs
(output) (input)

→ vector field: e.g. wind
2D input (location)
2D output (wind vector)



→ $f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ $f(\vec{x}) = \vec{y}$, $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$

$\vec{x}$ $\vec{y}$

DIFFRENTIATION:

single: $\frac{dy}{dx} = f'(x)$ $\Delta y \approx 6 \Delta x$ scaling factor

e.g. $y = x^2$ @ $x = 3$:
$f'(x) = 2x$, $f'(3) = \frac{dy}{dx}\Big|_{x=3} = \boxed{6}$

e.g. $3.001^2 \approx 9.006$
$2.999^2 \approx 8.994$

generalize to multi:
$\vec{y} = f(\vec{x})$, $\vec{x} \in \mathbb{R}^7$, $\vec{y} \in \mathbb{R}^4$
near $\vec{x} = \vec{a}$, fixed

given $\Delta\vec{x}$ (small changes in $\vec{x}$ near $\vec{a}$),
what's corresponding $\Delta\vec{y}$?

$\underset{4D}{\Delta\vec{y}} \approx \boxed{?} \underset{7D}{\Delta\vec{x}}$ ⇒ to convert 7D to 4D vector, use 4 x 7 matrix
row col

→ Ergo, we're going to do some linear algebra.

Vectors & Dot Product:

$\vec{v} \in \mathbb{R}^n$, n-tuple of real #'s

we can add & scale vectors.

Notations:

column vectors $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ or $\vec{v} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_1 \underset{\uparrow}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + x_2 \underset{\uparrow}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\uparrow}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$\vec{v} = \langle x_1, x_2, x_3 \rangle$

$= x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$

$= x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$, unit vectors

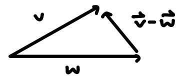
Dot Product (multiply vectors):

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \rightarrow \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$

turns out: dot product helps measure angle

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$, θ = angle b/t $\vec{v}$ & $\vec{w}$

Proof:

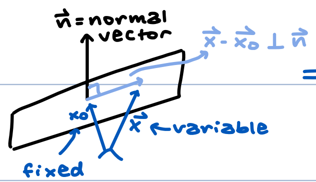


compute $|\vec{v} - \vec{w}|^2$

Note: $\vec{v} \perp \vec{w}$ iff $\vec{v} \cdot \vec{w} = 0$ (comes from law of cosines)

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

Plane Equations:



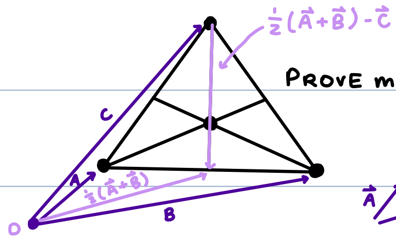
$\Rightarrow \vec{n} \cdot (\vec{x} - \vec{x_0}) = 0$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \vec{x_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

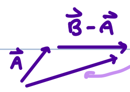
$\vec{x} - \vec{x_0} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ $\quad \vec{n} \cdot (\vec{x} - \vec{x_0}) = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

plane through (x_0, y_0, z_0) w/ normal $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

[2/5/25 - recitation]



Prove midpoints of a triangle are concurrent (meet @ same point)



$A + \frac{1}{2}(\vec{B} - \vec{A}) = \frac{1}{2}(\vec{A} + \vec{B})$

line through pt. $O + \vec{C}$ in direction $\frac{\vec{A} + \vec{B}}{2} - \vec{C}$ is $O + \vec{C} + t\left(\frac{\vec{A} + \vec{B}}{2} - \vec{C}\right)$ @ $t = \frac{2}{3}$, we're @ $O + \frac{\vec{A} + \vec{B} + \vec{C}}{3}$ ← symmetry!

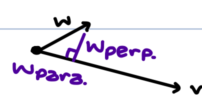
in general, line through point p in direction v is parameterized by $p + tv$

Dot Product: $\vec{v} = (v_1, \ldots v_n)$, $\vec{w} = (w_1, \ldots, w_n)$ then $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

Theorem: $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta$

→ if $\vec{v}$ & $\vec{w}$ are $\perp$, then $\vec{v} \cdot \vec{w} = 0$

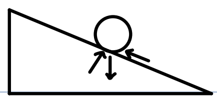
→ if $\vec{v}$ & $\vec{w}$ are same direction $\Rightarrow$ then $\vec{v} \cdot \vec{w} = |v| \cdot |w|$



$w = w^{perp} + w^{para}$

$v \cdot w = v \cdot w^{para} + \underbrace{v \cdot w^{perp}}_{0}$

$= |v| \cdot |w^{para}|$



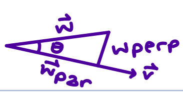
$w^{para} = a \cdot v$ for some a

then $v \cdot w^{para} = a|v|^2$

Q: given vectors $\vec{v}$ & $\vec{w}$, how to find decomposition into w^{para} & w^{perp}?

$v \cdot w = v \cdot w^{par}$ & we know $w^{par} = av$ for some real # a & $av \cdot v = a|v|^2$

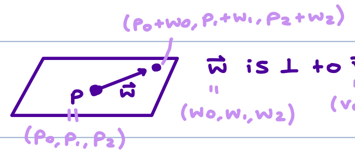
so $v \cdot w = a|v^2|$ so $a = \frac{v \cdot w}{|v|^2}$ so $w^{par} = \frac{v \cdot w}{|v|^2} \vec{v}$



then $w^{perp} = \vec{w} - w^{par}$

Dot products to parameterize planes:

Q: what is eqn. of plane through p. $\perp$ to vector $\vec{v}$?



$\vec{w}$ is $\perp$ to $\vec{v}$ $\quad \therefore \vec{w}\cdot\vec{v}=0$

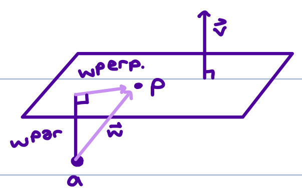
$\vec{w} = (w_0, w_1, w_2)$, $\vec{v} = (v_0, v_1, v_2)$

so (x_0, x_1, x_2) is on the plane if this is pt. $\vec{w}$ for some $\vec{w}$ with $\vec{w}\cdot\vec{v}=0$

$$(x_0-p_0, x_1-p_1, x_2-p_2)\cdot(v_0, v_1, v_2)=0$$

$$(x_0-p_0)v_0 + (x_1-p_1)v_1 + (x_2-p_2)v_2 = 0$$

Q: given a point $\vec{a}$, what's distance b/t. $\vec{a}$ & plane through $\vec{p}$ $\perp$ to $\vec{v}$



$$\vec{w}\cdot\vec{v} = w^{\text{par to } v}\cdot\vec{v} = |w^{par}|\cdot|v|^2$$

$w^{\text{par to } v} = \pm|w^{par}|.v$

so $|w^{\text{par. to } v}| = \pm\dfrac{\vec{w}\cdot\vec{v}}{|v|^2}$

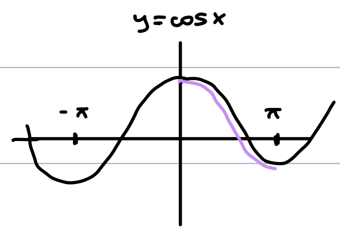
so distance from a to plane $= \left|\dfrac{\vec{w}\cdot\vec{v}}{|v|^2}\right|$

[2/6/25 - lecture 2] →
- projections
- cross products
- planes in 3D

RANDOM ITEMS: (clean-up)

→ if $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then $|\vec{v}| = \sqrt{v_1^2 + \dots v_n^2}$ → alt. notation: $\|\vec{v}\|$

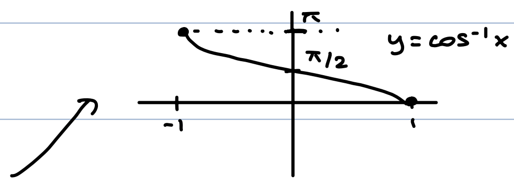
→ we say $\vec{u}$ is unit vector iff $|\vec{u}|=1$
- often write unit vectors as $\hat{u}$ ← hat notation



SPECIAL UNIT VECTORS: in 2D, $\hat{x} = \begin{bmatrix}1\\0\end{bmatrix}$, $\hat{y} = \begin{bmatrix}0\\1\end{bmatrix}$

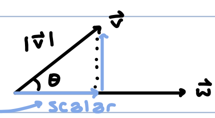
in 3D $\hat{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\hat{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

in n-D: $\hat{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, ..., $\hat{e}_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$



- from $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$, $\theta = \cos^{-1}\left(\dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)$ → gives unique values b/t 0 → π radians

- $\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 = |\vec{v}|^2$
- perpendicular, orthogonal, normal all mean same thing pretty much



PROJECTIONS: given $\vec{v}, \vec{w} \in \mathbb{R}^n$, the "component" of $\vec{v}$ along $\vec{w}$ is

formula is $|\vec{v}|\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = comp_{\vec{w}}(\vec{v})$ (could be negative)

"projection of $\vec{v}$ onto $\vec{w}$" = "component of $\vec{v}$ in direction of $\vec{w}$"

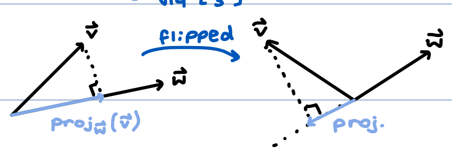
→ projection of $\vec{v}$ along $\vec{w}$ is the vector whose:
- direction is same as $\vec{w}$
- magnitude is component of $\vec{v}$ along $\vec{w}$

→ useful aside example: what's vector in direction of $\vec{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ but w/ magnitude 7?

* notice $\dfrac{\vec{w}}{|\vec{w}|}$ is unit vector $= \dfrac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$ ← scales down to 1

so vector we want is $\dfrac{7}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$ ← scales up to 7 (if opposite, flip sign)

PICTURES:



FORMULA: $Proj_{\vec{w}}(\vec{v})$ is vector in direction of $\vec{w}$ w/ (signed) "magnitude" $\dfrac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$

$$\boxed{proj_{\vec{w}}(\vec{v})} = \underbrace{\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}\right)}_{\text{magnitude}}\underbrace{\left(\frac{\vec{w}}{|\vec{w}|}\right)}_{\text{unit vector}} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w} = \boxed{\left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)}\vec{w}$$ ← scalar component

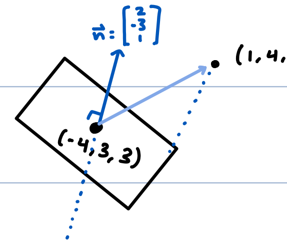
projecting $\vec{v}$ along $\vec{w}$

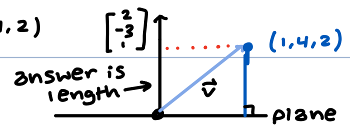
$$= \left(\vec{v}\cdot\left(\frac{\vec{w}}{|\vec{w}|}\right)\right)\hat{u} = \boxed{(\vec{v}\cdot\hat{u})\hat{u}}$$ ← if you "normalize" $\vec{w}$ to length 1

APPLICATION: distance from a point to a plane

pt. $(1,4,2)$ pt on plane $(-4,3,3)$

plane $2(x-4)-3(y-3)+(z-3)$





projecting light blue vector onto line containing $\vec{n}$

$\vec{v} = (1,4,2) - (-4,3,3) = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

$\text{comp}_{\vec{n}}(\vec{v}) = \dfrac{\vec{v}\cdot\vec{n}}{|\vec{n}|} = \boxed{\dfrac{6}{\sqrt{14}}}$

[LEC. 3 - 2/18/25] - CROSS PRODUCT OF VECTORS (only in 3D)

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (determinant)

$\vec{v}\times\vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - w_2 v_3 \\ -(v_3 w_1 - w_3 v_1) \\ v_1 w_2 - w_1 v_2 \end{bmatrix}$ 2 3 / 3 1 / 1 2

*remember to negate the ↑

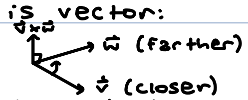
EX: $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (4)(2)-(1)(-1) \\ (-1)(3)-(2)(2) \\ (2)(1)-(3)(4) \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \\ -10 \end{bmatrix}$

EX: $\vec{v}\cdot(\vec{v}\times\vec{w}) = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 9 \\ -7 \\ -10 \end{bmatrix} = 18-28+10 = 0$

$\vec{w}\cdot(\vec{v}\times\vec{w}) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 9 \\ -7 \\ -10 \end{bmatrix} = 27-7-20 = 0$

$\vec{v}\times\vec{w}$ is $\perp$ to both $\vec{v}$ & $\vec{w}$!

GIVEN $\vec{v},\vec{w} \in \mathbb{R}^3$, cross product $\vec{v}\times\vec{w}$ is vector:

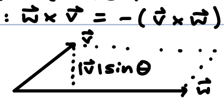


- **DIRECTION:** $\perp$ to both $\vec{v}$ × $\vec{w}$ & determined by "RHR"

(note: $\vec{w}\times\vec{v} = -(\vec{v}\times\vec{w})$)

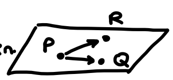
- **MAGNITUDE:** $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$

area of parallelogram



NOTICE: cross product can also help us find plane eqns:

3pts. determin plane



$\vec{PQ}\times\vec{PR} = \vec{n}$, normal to plane

LINE EQN. 2D	vs.	PLANE EQN 3D
$y = mx + b$ → slope-intercept		$z = mx + ny + b$
$y - y_0 = m(x - x_0)$ → pt-slope		$z - z_0 = m(x-x_0) + n(y-y_0)$
general: $a(x-x_0) + b(y-y_0) = 0$		$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$ general
$ax + by = c$		$ax + by + cz = d$

SYSTEMS OF LINEAR EQNS:

EX:
$3x - y - 2z = 1$
$x + y - z = -2$
$2x + 3y + z = 4$

$x\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + z\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ $\xrightarrow{\text{matrix form}}$ $\underbrace{\begin{bmatrix} 3 & -1 & -2 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}}_{\vec{b}}$ $A\vec{x} = \vec{b}$

"solving linear combos of vectors"

aug. mat. $\left[\begin{array}{ccc:c} 3 & -1 & -2 & 1 \\ 1 & 1 & -1 & -2 \\ 2 & 3 & 1 & 4 \end{array}\right]$ $\xrightarrow[\text{"row reduc."}]{\text{"gaussian elim"}}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

coefficient matrix   augmented column

$x + 0y + 0z = 2$
⋮
→ $x = 2$, $y = -1$, $z = 3$

MATRIX MULTIPLICATION:

$2x_1 + 3x_2 = y_1$ $4x_1 + 5x_2 = y_2$	$\underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}}$
$6y_1 + 7y_2 = z_1$ $8y_1 + 9y_2 = z_2$	$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$B\vec{x} = \vec{y}$, $A\vec{y} = \vec{z}$ → $A(B\vec{x}) = \vec{z}$ $(AB)\vec{x} = \vec{z}$

plug in: $6(2x_1 + 3x_2) + 7(4x_1 + 5x_2) = z_1$ → $(6\cdot2 + 7\cdot4)x_1 + (6\cdot3 + 7\cdot5)x_2 = z_1$

$AB = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6\cdot2 + 7\cdot4 & 6\cdot3 + 7\cdot5 \\ 8\cdot2 + 9\cdot4 & 8\cdot3 + 9\cdot5 \end{bmatrix}$

[2/10/25 · rec]

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

DOT PRODUCT vs. CROSS PRODUCT

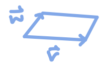
Dot product

$\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cos\theta$

- $\vec{v}\cdot\vec{w}$ is a #
- $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
- $(v_1,\dots,v_n)\cdot(w_1,\dots,w_n) = v_1w_1 + \dots + v_nw_n$
- well defined in any direction

Cross product

$\vec{v}\times\vec{w} = \underbrace{|\vec{v}|\cdot|\vec{w}|\sin\theta}_{\text{area of parallelogram}}\,\hat{n}$



- cross product $\vec{v}\times\vec{w}$ is a vector
- $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$ (direction flips - RHR)
- $(v_1,v_2,v_3)\times(w_1,w_2,w_3) = (v_2w_3 - v_3w_2,\ -v_1w_3 + v_3w_1,\ v_1w_2 - v_2w_1)$
- only defined on $\mathbb{R}^3$

$$\left(\vec{v}\times\vec{w} = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right)$$

ROW REDUCTION:

Ex: $3x - y + 2z = 1$
$x + y + z = -2$
$2x + 3y + z = 4$

$$\longrightarrow \begin{bmatrix} 3 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{3}\text{row 1} + \text{row 2}} \begin{bmatrix} 3 & -1 & 2 & 1 \\ 0 & \frac{4}{3} & \frac{1}{3} & -\frac{7}{3} \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

⋮
etc.

$$\xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}$$

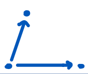
3 moves you can do to matrix to reduce it to this simple form:

1) swap rows
2) multiply a row by nonzero constant
3) add scalar multiple of a row to another row

PRACTICE:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1) Find eqn. of plane through $(2,1,1)$ $(0,4,2)$ and $(-1,3,1)$



2 vectors:

$\langle -1,3,1\rangle - \langle 0,4,2\rangle = \langle -1,-1,-1\rangle$

$\langle 2,1,1\rangle - \langle 0,4,2\rangle = \langle 2,-3,-1\rangle$

normal vector:

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & -1 \\ 2 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 1-3 \\ -(1--2) \\ 3--2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix}$$

plane eqn: $a(x-x_0) + b(y-y_0) + c(z-z_0) = d$

pt: $(0,4,2)$
a, b, c : $\langle -2,-3,5\rangle$
$\rightarrow -2(x-0) - 3(y-4) + 5(z-2) = 0$

$-2x - 3y + 12 + 5z - 10 = 0$

$$\boxed{-2x - 3y + 5z = -2}$$

b) find area of triangle formed by connecting pts

parallelogram area $= |\vec{v}|\cdot|\vec{w}|\sin\theta\,\hat{n} = \left(\sqrt{-1^2 + -1^2 + -1^2}\right)\left(\sqrt{2^2+3^2+1^2}\right)\sin\theta\left(\frac{\langle -2,-3,5\rangle}{\sqrt{2^2+3^2+5^2}}\right)$

$= (\sqrt{3})(\sqrt{14})\sin\theta\left(\frac{\langle -2,-3,5\rangle}{\sqrt{38}}\right) = \frac{\sqrt{42}}{\sqrt{38}}\sin\theta\,\langle -2,-3,5\rangle$

2) row reduct. (ex. above)

3) when does
$ax + by = r$
$cx + dy = s$
have zero solutions? 1 solution?

* $Ax = 0$ have nonzero solution when $\det|A| = 0$

[2/11/25] - LECTURE
- matrix-matrix & matrix-vector multiplication
- linear transformations
- geometric transformations (rotations, reflections, projections)

$B\vec{x} = \vec{y} \rightsquigarrow A\vec{y} = \vec{z} \longrightarrow A(B\vec{x}) = \vec{z} \to (AB)\vec{x} = \vec{z}$

LAST TIME: $b_{11}x_1 + b_{12}x_2 = y_1$ $\quad$ $a_{11}y_1 + a_{12}y_2 = z_1$

$a_{i,j}$ = entry in ith row, jth column

$b_{21}x_1 + b_{22}x_2 = y_2$ $\quad$ $a_{21}y_1 + a_{22}y_2 = z_2$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ $\quad$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

"compose": $(a_{11}b_{11} + a_{12}b_{21})x_1 + (a_{11}b_{12} + a_{12}b_{22})x_2 = z_1$

$(a_{21}b_{11} + a_{22}b_{21})x_1 + (a_{21}b_{12} + a_{22}b_{22})x_2 = z_2$

R = row
C = col

$$C = AB = \begin{array}{c} \begin{array}{cc} C_{11} & C_{12} \end{array} \\ \begin{bmatrix} (R1\,A)\cdot(C1\,B) & (R1\,A)\cdot(C2\,B) \\ (R2\,A)\cdot(C1\,B) & (R2\,A)\cdot(C2\,B) \end{bmatrix} \\ \begin{array}{cc} C_{21} & C_{22} \end{array} \end{array}$$

GENERALIZE: $Ay = \vec{z}$, A has m rows & n columns "m × n matrix"
(m z's) (n y's)
m eq's in n variables

$B\vec{x} = \vec{y}$, B has n rows & p columns, "n by p matrix"
n eqns in p variables
(n y's) (p x's)

$C = AB$ $\quad$ c_{ij} = (ith row of A) · (jth column of B)
(# cols of A) must equal (# row of B)

EX: (5×3 matrix)(3×6 matrix) = (5×6 matrix)
(3×6 matrix)(5×3 matrix) = X NO!

NOTE: A is n×n, $\vec{x}$ is m×1 matrix (= vector!)
$A\vec{x}$ is n×1 matrix (= vector!)

LINEAR TRANSFORMATIONS: $f: \mathbb{R} \to \mathbb{R}$ "f distributes our addition"
$y = f(x)$ is linear iff $f(x_1 + x_2) = f(x_1) + f(x_2)$ for any $x_1, x_2 \in \mathbb{R}$.

$f(x) = x^2$ NO $\quad$ $f(x) = mx + b$ NO. only if $b=0$ $\quad$ $f(x_1+x_2) = m(x_1+x_2)+b$
$f(x_1)+f(x_2) = mx_1 + b + mx_2 + b$

$f(x) = x$ YES.

the only such functions are $f(x) = ax$ for constant a
$f(x) = 5x \to f(3x) = 5(3x) = 3\cdot 5x = 3\cdot f(x)$

GENERALIZE: $\vec{y} = T(\vec{x})$, $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$

NOTATION: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
set of inputs (domain) $\quad$ set of outputs (codomain)

$$\boxed{\begin{array}{l} T \text{ is linear iff} \\ T(a\vec{x}_1 + b\vec{x}_2) = a\cdot T(\vec{x}_1) + b\cdot T(\vec{x}_2) \end{array}}$$

TURNS OUT: T is linear iff $T(\vec{x}) = A\vec{x}$ for some m×n matrix A.

EX: $T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad$ $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

write $1x + 1y + 0z = w_1$
$0x + 0y + 1z = w_2$ $\quad \rightsquigarrow$ so $T(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
(2×3 matrix)

check: $A\vec{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ z \end{bmatrix}$ ✓

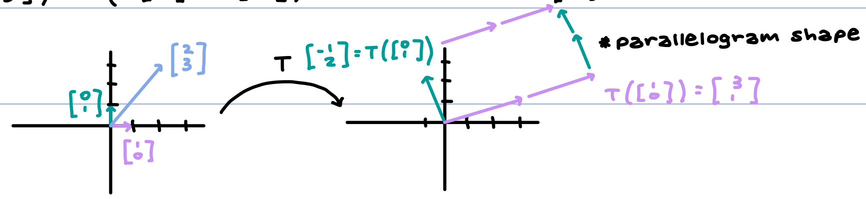
EX: $T: \mathbb{R}^2 \to \mathbb{R}^2$, linear

say we know: $T(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\hat{\imath}}) = \begin{bmatrix}3\\1\end{bmatrix}$ & $T(\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\hat{\jmath}}) = \begin{bmatrix}-1\\2\end{bmatrix}$

Q: what is $T(\begin{bmatrix}2\\3\end{bmatrix})$?

$$T(\begin{bmatrix}2\\3\end{bmatrix}) = T(2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}) = T(2\hat{\imath} + 3\hat{\jmath}) = 2\begin{bmatrix}3\\1\end{bmatrix} + 3\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$$

VISUALIZE:



MATRIX: $T(\begin{bmatrix}x\\y\end{bmatrix}) = T(x\hat{\imath} + y\hat{\jmath}) = x \cdot T(\hat{\imath}) + y \cdot T(\hat{\jmath}) = x\begin{bmatrix}3\\1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix}$

$$= \begin{bmatrix}3x - y\\x + 2y\end{bmatrix} = \begin{bmatrix}3 & -1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

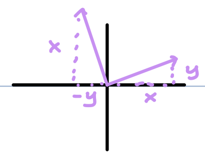
columns are $T(\begin{bmatrix}1\\0\end{bmatrix})$, $T(\begin{bmatrix}0\\1\end{bmatrix})$

* in general, $T(\vec{x}) = A\vec{x}$, linear A is the matrix whose columns are $T(\hat{e}_1), T(\hat{e}_2), \dots, T(\hat{e}_n)$

EX: $T: \mathbb{R}^2 \to \mathbb{R}^2$, rotation 90° CCW

$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$ matrix is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}-1\\0\end{bmatrix}$ $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$



[OFFICE HRS:] - 2/12/25

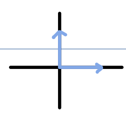
$I = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \cdot \begin{pmatrix}m_{11} & m_{12} & \cdots\\ \vdots & \ddots & \\ \vdots & & \end{pmatrix} = \begin{pmatrix}m_{11} & m_{12} & \cdots\\ \vdots & \ddots & \\ \vdots & & \end{pmatrix}$ ← same!

given square matrix A, is there another matrix so that $AA^{-1} = I$

A^{-1} exists $\longleftrightarrow$ $\det A \neq 0$

$(A)(x) = (b) \rightsquigarrow A^{-1}Ax = Ix = x = A^{-1}b$ ← so this has a unique solution

EX: $\begin{pmatrix}1\\0\end{pmatrix}$ rotated 90° ccw $\rightsquigarrow \begin{pmatrix}0\\1\end{pmatrix}$



$\begin{pmatrix}0\\1\end{pmatrix}$ rotated 90° ccw $\rightsquigarrow \begin{pmatrix}-1\\0\end{pmatrix}$

then rotating $\begin{pmatrix}a\\b\end{pmatrix}$ 90° ccw is rotating $a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \to T(a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix})$

$$= T(a\begin{pmatrix}1\\0\end{pmatrix}) + T(b\begin{pmatrix}0\\1\end{pmatrix})$$

$$= aT\begin{pmatrix}1\\0\end{pmatrix} + bT\begin{pmatrix}0\\1\end{pmatrix}$$

DETERMINANT:

EX: $\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ doesn't have inverse $\Rightarrow \det = 0$ or "rows aren't independent"

$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}c & d\\0 & 0\end{pmatrix}$

$\det\begin{pmatrix}a & b\\c & d\end{pmatrix} = ad - bc$

ROW OP:

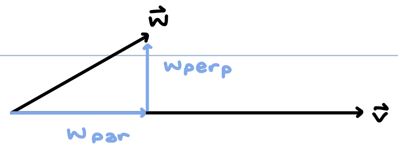
$$\left(\begin{array}{cc|cc}a & b & 1 & 0\\c & d & 0 & 1\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & \text{inverse}\\0 & 1 & \text{matrix}\end{array}\right)$$

DET: 3x3: "cofactor"

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} & \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ & & \\ & & \end{bmatrix}$$

$\det = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

PROJECTION VS. COMPONENT:



projection onto a line: linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$

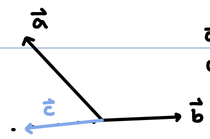
find component $\vec{w}$ parallel to $\vec{v}$: $\vec{w}_{par}$

find scalar component of $\vec{w}$ parallel to $\vec{v}$: $|\vec{w}_{par}|$

$$\vec{v} \times \vec{w} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \hat{\imath}\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - \hat{\jmath}\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + \hat{k}\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

$$= \begin{pmatrix} v_2w_3 - v_3w_2 \\ -(v_1w_3 - v_3w_1) \\ v_1w_2 - v_2w_1 \end{pmatrix} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ -v_1w_3 + v_3w_1 \\ v_1w_2 - v_2w_1 \end{pmatrix}$$

EX:



$\vec{a} = \vec{c} + \vec{d}$

$\text{comp}_b\, a = -3$ ← meaning, must be in − direction compared to $\vec{b}$

[REC] - 2/12/25 - MATRIX MULTIPLICATION:

IDENTITY MATRIX (of dim. n): $\overset{(n\times n)}{\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & & & 1 \end{bmatrix}} = I_n$

→ if M is a matrix such that I_nM is well-defined, then $I_nM = M$ & similarly $MI_n = M$ if well-defined

INVERSES: if M is a $n\times n$ square matrix, is there a matrix M^{-1} such that $MM^{-1} = M^{-1}M = I_n$

→ answer: sometimes. can tell by whether the determinant $= 0$

- related to system of linear eqns. in following way: can encode n eqns in n vars: $\underset{n\times n}{M}\,\underset{n\times 1}{x} = \underset{n\times 1}{b}$

Suppose M has inverse M^{-1}. then $M^{-1}Mx = M^{-1}b$, where $M^{-1}Mx = I_nx = x$

Some get $x = M^{-1}b$. this is unique solution to our system.

EX: system of 2 eqns in 2 vars $\longleftrightarrow$ 2 lines in plane $\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$

solutions $\longleftrightarrow$ intersection pts of 2 lines (0, 1, or ∞ pts)

unique solutions $\longleftrightarrow$ 2 lines aren't parallel

the determinent gives a formula for testing whether M has inverse $\longleftrightarrow$ $Mx = b$ has unique solution $\longleftrightarrow$ 2 lines aren't parallel

in our example, the lines are parallel exactly when (c, d) is multiple of (a, b) → $ad - bc = 0$

we define determinant of 2x2 matrix to be $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

then this matrix has an inverse $\longleftrightarrow$ $\det M \neq 0$

Ex: suppose I have function $T: \mathbb{R}^n \to \mathbb{R}^m$. what properties must T satisfy to be a linear trans?

- $T(a+b) = T(a) + T(b)$
- $T(ca) = cT(a)$ for $c \in \mathbb{R}$

these two properties show that T is determined by a matrix.

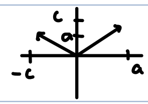
$$T\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = T\left(a_1\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} + \dots + a_n\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}\right) = a_1 T\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} + a_2 T\begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} + \dots + a_n\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

matrix encodes $T\begin{pmatrix} 1 \\ \vdots \end{pmatrix}, \dots T\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ and then matrix mult. tells how to compute $T\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

QUESTIONS: find matrices for the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$

1. rotate 90° ccwise around origin
2. reflect around x-axis
3. rotate θ radians CCW around origin
4. project onto x-axis
5. scale by factor of 3
6. project onto line spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

1. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} \to \begin{bmatrix} -c \\ a \end{bmatrix}$



2. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} \to \begin{bmatrix} a \\ -c \end{bmatrix}$
3. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} \to \begin{bmatrix} a\cos\theta + c\sin\theta \\ a\sin\theta - c\cos\theta \end{bmatrix}$
4. $\text{proj}_x \vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$
5. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
6. $\text{proj}_{\ell} \vec{v} = \dfrac{\vec{v}\cdot\vec{\ell}}{\vec{\ell}\cdot\vec{\ell}}\vec{\ell}$

[2/13/25] - lecture - classic linear algebra
- linear combos/span
- linear independence
- ~~bases~~

LINEAR COMBINATIONS: given $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}$, we can add/scale them to make new vectors.

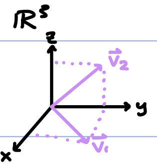
Ex: $2\vec{v}_1 + 5\vec{v}_2 - \pi\vec{v}_3$ ← linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

Ex: every $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ ← every 3D $\vec{v}$ is a combo of $\hat{\imath}, \hat{\jmath}, \hat{k}$

GIVEN $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$, we define span$\{\vec{v}_1, \dots, \vec{v}_k\}$ = set of all linear combos of $\vec{v}_1, \dots, \vec{v}_k$ = $\{c_1\vec{v}_1 + c_2\vec{v}_2 \dots\}$ where $c_1, \dots, c_k$ are scalars

Ex: span$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ = all of $\mathbb{R}^3$

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$



span$\{\vec{v}_1, \vec{v}_2\}$ = plane in $\mathbb{R}^3$ through origin = plane whose eqn. is $x - y + z = 0$ ← can get from $\vec{v}_1 \times \vec{v}_2$

IN $\mathbb{R}^3$, span$\{\dots\}$ can be:

3D ① all of $\mathbb{R}^3$

2D ② plane through origin

1D ③ line through origin

0D ④ $\{\vec{0}\}$ ex: span$\{\vec{0}, \vec{0}, \vec{0}\}$

"subspaces of $\mathbb{R}^3$"

IS span$\{\vec{v}_1, \vec{v}_2\}$ always a plane?

→ NO. $\vec{v}_1$ could $= \vec{v}_2$

ex: span$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}\right\}$ = span$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ = line through origin in direction $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

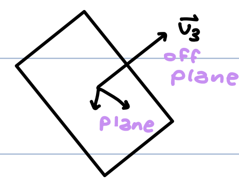
is span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ always all of $\mathbb{R}^3$?

→ NO. span could be a line again, or a plane.

ex: span$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$ = span$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

b/c $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

but if span$\{\vec{v}_1, \vec{v}_2\}$ is a plane & $\vec{v}_3 \notin$ span$\{\vec{v}_1, \vec{v}_2\}$, then span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = all of $\mathbb{R}^3$



LINEAR INDEPENDENCE of vectors:

a set of vectors $\{\vec{v}_1, \dots \vec{v}_k\}$ is linearly dependent iff one is a linear combo. of (some of) the others.

MORE FORMALLY: there are scalars $c_1, c_2 \dots c_k$ that are not all 0 such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$

Ex: $\left\{\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}\right\}$ $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$ OR $2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$

Ex: $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$ $\vec{v}_2 = 5\vec{v}_1$ OR $5\vec{v}_1 - \vec{v}_2 + 0\vec{v}_3 = \vec{0}$

LINEARLY INDEPENDENT: NOT linearly dependent.

→ $\{\vec{v}_1, \dots, \vec{v}_k\}$ is lin. ind. iff $c_1\vec{v}_1 + \dots + c_k v_k = \vec{0}$ → $c_1 + \dots + c_k = 0$

EX: $\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

→ $\begin{bmatrix}c_1\\c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\c_2\end{bmatrix} = \begin{bmatrix}c_1\\c_1+c_2\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

but if $\vec{v}_2 = \begin{bmatrix}5\\5\\0\end{bmatrix}$ instead, $c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix}c_1+5c_2\\c_1+5c_2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ → $c_1 = 5$, $c_2 = -1$

how to tell if set is linearly independent?

e.g. $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}5\\2\\-3\end{bmatrix}\right\}$ consider $c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}5\\2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

→ $\begin{bmatrix}1&0&5\\1&1&2\\0&1&-3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ OR solve $\left[\begin{array}{ccc|c}1&0&5&0\\1&1&2&0\\0&1&-3&0\end{array}\right]$

$\left[\begin{array}{ccc|c}1&0&5&0\\1&1&2&0\\0&1&-3&0\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c}1&0&5&0\\0&1&-3&0\\0&1&-3&0\end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c}1&0&5&0\\0&1&-3&0\\0&0&0&0\end{array}\right]$

$c_1 + 5c_3 = 0$

$c_2 - 3c_3 = 0$

$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-5c_3\\3c_3\\c_3\end{bmatrix} = c_3\begin{bmatrix}-5\\3\\1\end{bmatrix}$

$-5\begin{bmatrix}1\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}5\\2\\-3\end{bmatrix} = \vec{0}$

linearly dependent.

replace $\begin{bmatrix}5\\2\\-3\end{bmatrix}$ w/ $\begin{bmatrix}5\\2\\0\end{bmatrix}$

$c_1 + 5c_3 = 0$

$c_2 - 3c_3 = 0$

$c_3 = 0$ → forces $c_3 = 0$, $c_1 = 0$, $c_2 = 0$

∴ linearly independent

[2/14/25] - lecture
- bases
- determinant
- matrix inverses

LAST TIME: given a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$

- span = set of linear combos of vectors
- linear dependence/independence: is one vector linear combo of others?

in general, dim span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \begin{cases} <k \text{ if vectors dependent} \\ =k \text{ if vectors independent}\end{cases}$

given collection $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k \in \mathbb{R}^n$, we say they form a **basis** of $\mathbb{R}^n$ iff:

- span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathbb{R}^n$
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent

Ex: $\hat{i}, \hat{j}, \hat{k}$ in $\mathbb{R}^3$

$\left\{\begin{bmatrix}3\\5\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}23\\\pi\\0\end{bmatrix}\right\}$ NOT basis (stuck in xy plane)

$\left\{\begin{bmatrix}3\\5\\3\end{bmatrix}, \begin{bmatrix}-1\\17\\0\end{bmatrix}, \begin{bmatrix}23\\\pi\\7\end{bmatrix}\right\}$ is basis: row reduction $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

any basis of $\mathbb{R}^n$ must have n vectors. (use this to define "dimension")

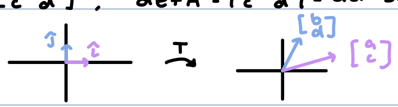
in $\mathbb{R}^n$, n = smallest # of vectors needed to span $\mathbb{R}^n$

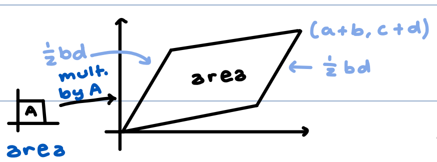
n = largest # of lin. ind. vectors

DETERMINANT: 2x2. $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, $\det A = \begin{vmatrix}a&b\\c&d\end{vmatrix} = ad - bc$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{x}) = A\vec{x}$





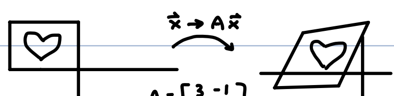
AREA = $(a+b)(c+d) - ad - bd - 2bc$

$= ac + bc + cd - ad - bd - 2bc$

$= ad - bc = \det A$

area scaling factor T(unit sq.)

det A = "area scaling factor"



$A = \begin{bmatrix}3&-1\\2&4\end{bmatrix}$

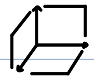
det A = 14

area = 14(3π)

3x3 DETERMINANT:

"volume scaling factor" for $\vec{x} \to A\vec{x}$

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$



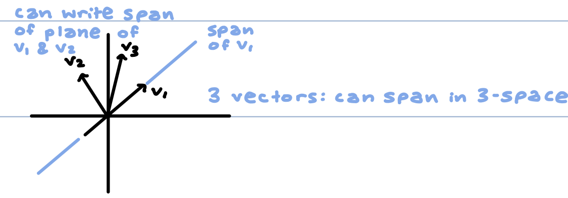
vol = 1

Suppose $\begin{bmatrix} a \\ d \\ g \end{bmatrix}, \begin{bmatrix} b \\ e \\ h \end{bmatrix}, \begin{bmatrix} c \\ f \\ i \end{bmatrix}$ linearly dependent

→ coplanar (i.e. on same plane) ⇒ parallelpiped would have value 0 ⟷ $\det A = 0$

↳ 3D analog of parallelogram

[2/18/25] - recitation



SPAN: what is the span of $\{v_1, \dots, v_m\}$ in $\mathbb{R}^n$?

→ subspace of $\mathbb{R}^n$ consisting of all linear combos of vectors v_i

Ex: $\hat{i}$ & $\hat{j}$ span $\mathbb{R}^2$.
$\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hat{i}, \hat{j}, \hat{k}$ span $\mathbb{R}^3$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ doesn't span $\mathbb{R}^3$, only spans a plane ($\mathbb{R}^2$) b/c $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is linear combo.

NOTICE: n vectors span a space of dimension ≤ n (expect dim. n, but there could be "redundancies")

LINEAR INDEPENDENCE: what does it mean to say $\{v_1, \dots, v_n\}$ in $\mathbb{R}^n$ are linearly independent?

- no vector in the set is a linear combo. of the others. in other words, if $a_1v_1 + a_2v_2 + \dots + a_mv_m = 0$, then all $a_i = 0$
- in other words, no v_i is in the span of the other v_i's, so $\{v_1, \dots, v_m\}$ span a subspace of dimension m. in particular, $m \le n$

Ex: $\hat{i}$ & $\hat{j}$ are linearly ind. in $\mathbb{R}^2$
$\hat{i}, \hat{j}, \hat{k}$ linearly ind. in $\mathbb{R}^3$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are not linearly ind.

BASIS: set of vectors $\{v_1, \dots, v_m\}$ in $\mathbb{R}^n$ is basis if they are linearly independent and span $\mathbb{R}^n$.
NOTE: every basis of size n - this will define dimension of general vector spaces.

- Ex: $\{\hat{i}, \hat{j}, \hat{k}\}$ is a basis for $\mathbb{R}^3$
- Ex: $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is basis for $\mathbb{R}^n$

HOW TO TELL if n vectors form a basis?

DETERMINANTS:

- det M is calculated via recursive formula
  - det(a) (1x1 matrix) = a
  - det(b) (2x2 matrix) = $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = +a \cdot \det(d) - b \cdot \det(c) = ad - bc$
  - $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix} = a(ei - fh) - b(di - fg) + c(dh - ge)$

pick row or col & expand along that $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

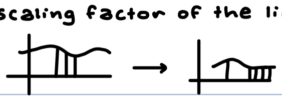
RELATIONSHIP TO INVERSE MATRIX:

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when determinant ≠ 0 & there's no inverse when det = 0 (for any dim.)

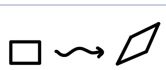
↳ can be generalized to higher dimension

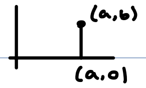
- det M is the area scaling factor of the linear transformation M

$u = f(x)$
$du = f'(x)dx$



$\det(v_1 | \dots | v_n) = 0$ exactly when $v_1, \dots, v_n$ don't form basis of $\mathbb{R}^n$



Ex: 

(y-pts go to origin)

project onto x-axis linear transformation T given by $\begin{pmatrix}1&0\\0&0\end{pmatrix}$

check: $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ ✓

$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ ✓ and since $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ form a basis, i know this is right matrix.

$\det\begin{pmatrix}1&0\\0&0\end{pmatrix}=0$

*when det = 0, no basis is formed.

$T\begin{pmatrix}r\\s\end{pmatrix}=T\left(r\begin{pmatrix}1\\0\end{pmatrix}+s\begin{pmatrix}0\\1\end{pmatrix}\right)=rT\begin{pmatrix}1\\0\end{pmatrix}+sT\begin{pmatrix}0\\1\end{pmatrix}$

[2/19/25] - office hours & rec:

LINEAR TRANSFORMATION: $M_2 M_1 \begin{pmatrix}a\\b\end{pmatrix}$

$$T\begin{pmatrix}a\\b\end{pmatrix}=T\left(a\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}0\\1\end{pmatrix}\right)\overset{\text{T is a linear transformation, so } T(v+w)=Tv+Tw}{=}T\left(a\begin{pmatrix}1\\0\end{pmatrix}\right)+T\left(b\begin{pmatrix}0\\1\end{pmatrix}\right)\overset{T(cv)=cT(v)}{=}aT\begin{pmatrix}1\\0\end{pmatrix}+bT\begin{pmatrix}0\\1\end{pmatrix}$$

what happens to (1,0) | what happens to (0,1)

$R_{\frac{\pi}{2}}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ $R_\theta\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\rightarrow\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

PLANE EQUATION: $ax+by+cz=d$ ← forms a plane w/ normal vector $\vec{n}=(a,b,c)$

- (x_2,y_2,z_2) & (x_3,y_3,z_3) on the plane ⟶ $(x_2-x_3, y_2-y_3, z_2-z_3)$ is vector b/t them
- $(a,b,c)\cdot(x_2-x_3,y_2-y_3,z_2-z_3)=ax_2+by_2+cz_2-ax_3-by_3-cz_3=d-d=0$

EIGENVECTORS & EIGENVALUES:

Ex: google Taylor Swift — what is the best website?
→ what would happen if started in a random place & kept following links?
let's describe this with a matrix:

$$\begin{array}{c|cccc} & \text{julie} & \text{wiki} & \text{sally} & \text{official} \dots \\ \text{julie} & 0 & 1/3 & 1/3 & 1/3 \\ \text{wiki} & 0 & 0 & 0 & 1 \\ \text{sally} & 3/5 & 0 & 0 & 1/5 \\ \text{official} & & & & \\ \vdots & & & & \end{array}$$

clicking on bunch of random links ⟷ multiplying matrix by bunch of times

how to figure out what n high power of this matrix looks like?

if v is vector such that $Av=17v$, then $A^n v=17^n v$. such a v is called an eigenvector w/ eigenvalue 17.

if we can understand eigenvectors, then we can compute high powers of the matrix easily.

$A\vec{v}=\lambda\vec{v}$

Ex: population of rabbits & foxes

$r(n)$ = # rabbits in year n
$f(n)$ = # foxes in year n] in year 0, there are 30 rabbits & 20 foxes

$r(n+1)=4r(n)-2f(n)$
$f(n+1)=r(n)+f(n)$

1) what do the coefficients mean? do they make sense?
yes. relationship b/t rabbit & fox pop.

2) how many foxes & rabbits are there after 1 yr? 2 yrs? n?

$r(0)=30$ $r(1)=4(30)-2(20)=80$ $r(2)=4(80)-2(50)=220$
$f(0)=20$ $f(1)=30+20=50$ $f(2)=50+80=130$

let $\vec{v}=\begin{bmatrix}10\\10\end{bmatrix}$ & $\vec{w}=\begin{bmatrix}20\\10\end{bmatrix}$. call this matrix A.

what is Av? A^2v? A^nv? what about w? → $2^n\begin{bmatrix}10\\10\end{bmatrix}$

$\begin{bmatrix}4&-2\\1&1\end{bmatrix}\begin{pmatrix}30\\20\end{pmatrix}\longrightarrow\begin{bmatrix}4&-2\\1&1\end{bmatrix}\begin{pmatrix}80\\50\end{pmatrix}\longrightarrow$ etc.

write original vect. as lin. combo. of v & w $c_1\begin{bmatrix}10\\10\end{bmatrix}+c_2\begin{bmatrix}20\\10\end{bmatrix}=\begin{bmatrix}30\\20\end{bmatrix}$, c_1 & $c_2=1$

what is eventual ratio of rabbits to foxes? 2:1

how to find $\vec{v}$ & $\vec{w}$?

[2/20/25] - lecture
- eigenvectors/eigenvalues
- diagonalization

RECALL **theorem**: let A be $n\times n$ matrix. the following are equivalent:

1) $\det A \neq 0$
2) columns of A form basis of $\mathbb{R}^n$
3) columns of A span $\mathbb{R}^n$
4) columns of A are linearly independent
5) linear system $A\vec{x}=\vec{b}$ has unique solution ← for any choice of $\vec{b}$
6) A^{-1} exists $(AA^{-1} = A^{-1}A = I_n)$ → $\vec{x} = A^{-1}\vec{b}$

$A\vec{x}=\vec{b}$
$AA^{-1}\vec{x} = A^{-1}\vec{b}$
$\vec{x} = A^{-1}\vec{b}$

RECALL **theorem**: let A be $n\times n$ matrix. the following are equivalent:

1) $\det A = 0$
2) columns of A don't form basis of $\mathbb{R}^n$
3) columns of A don't span $\mathbb{R}^n$
4) columns of A are not linearly independent
5) linear system $A\vec{x}=\vec{b}$ don't has unique solution
6) A^{-1} not exists $(AA^{-1} = A^{-1}A = I_n)$

HOMOGENEOUS SYSTEMS: $A\vec{x}=\vec{0}$ ← always has solution $\vec{x}=0$

- always has solution $\vec{x}=\vec{0}$
  - $\det A \neq 0 \leftrightarrow \vec{0}$ only solution
  - $\det A = 0 \leftrightarrow$ there exists non-zero solutions

EIGENVECTORS & EIGENVALUES: $n\times n$ matrix A

we say nonzero $\vec{v}\in\mathbb{R}^n$ is <u>eigenvector</u> with <u>eigenvalue</u> λ (for A)
iff $A\vec{v}=\lambda\vec{v}$. $\vec{v}$ = eigenvect., λ = eigenval.
A scales $\vec{v}$ but doesn't change its direction

EX: $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = A$ notice: $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_2} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

*$\vec{0}$ is <u>never</u> considered an eigenvector

where does this come from? { so, $\vec{v}_1$ is eigenvector with eigenvalue 2, and $\vec{v}_2$ is eigenvector with eigenvalue 3.

if we write $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$ (we want to find x, y, λ)

$$\begin{aligned} 4x - 2y &= \lambda x \\ x + y &= \lambda y \end{aligned} \longrightarrow \left.\begin{aligned} (4-\lambda)x - 2y &= 0 \\ x + (1-\lambda)y &= 0 \end{aligned}\right]$$ homogeneous! we want nonzero solutions

$$\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← *has nonzero solns. when $\det = 0$

$\det = (4-\lambda)(1-\lambda) - (-2) = 0$

$4 - 5\lambda + \lambda^2 + 2 = 0$] characteristic equation

$(\lambda-3)(\lambda-2) = 0$

$\lambda = 3, 2$ ← eigenvalues!

in general, eigenvalues are solutions to: $\det(A-\lambda I) = 0$ ← characteristic polynomial, degree n

$I = \begin{bmatrix} 1 & 0 & \cdots \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$ identity matrix find eigenvectors associated to $\lambda=2$: plug $\lambda=2$ back in: $\begin{bmatrix} 4-2 & -2 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{array}{l} 2x-2y=0 \\ x-y=0 \end{array}\right\} \to x=y \to$ say $x=y=1$.

so $\vec{v}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

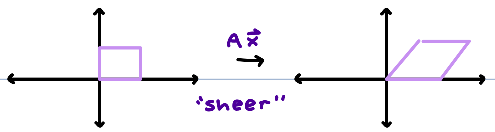
*try same thing for $\lambda=3$ to find eigenvector $\vec{v}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

in this example, eigenvectors form a basis of $\mathbb{R}^2$ (eigenbasis)
- not true in general

EX: $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to$ characteristic poly. $= \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0 \to \lambda=1$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow$ every vect. is eigenval. $\mathbb{R}^2$ — different.

plug in $\lambda=1 \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \to y=0$. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is only eigenvector



"wind blew over"

DIAGONALIZATION: $n\times n$ matrix A. Suppose A has eigenbasis: $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

let B = matrix whose columns are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

let $D=\begin{bmatrix} \lambda_1 & \cdots & \cdots \\ 0 & \lambda_2 & \cdots \\ 0 & 0 & \lambda_3 \cdots \end{bmatrix}$

then, $A=BDB^{-1}$ $\leftarrow$ diagonalize matrix

EX: $A=\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \underset{B}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}\underset{D}{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}\underset{B^{-1}}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}} \to \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}\frac{1}{2-1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ ✓

$A^4 = (BDB^{-1})(BDB^{-1})(BDB^{-1})(BDB^{-1})$
$= (BD\cancel{B^{-1}})(\cancel{B}DB^{-1})(BD\cancel{B^{-1}})(\cancel{B}DB^{-1}) = BD^4B^{-1}$

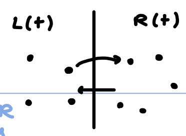
$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}^{100} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 2^2 \end{bmatrix}$

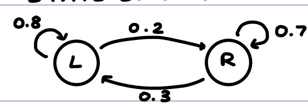
*read takeaways [2/21/25 - lecture] $\left\{\begin{array}{l} \cdot\text{diagonalization} \\ \cdot\text{polar coords.} \\ \cdot\text{complex number } \mathbb{C} \end{array}\right.$

$B=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

DIAGONALIZATION EX: markov chains



STATE DIAGRAM:



"transition probabilities"

80% L stay 30% R stay

$\left.\begin{array}{l} L(n+1) = 0.8L(n) + 0.3R(n) \\ R(n+1) = 0.2L(n) + 0.7R(n) \end{array}\right]$ add to 1

Q: long-term behavior of system?

$\begin{bmatrix} L(n+1) \\ R(n+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} L(n) \\ R(n) \end{bmatrix}$

EIGENVALUES: $\lambda=1, \lambda=0.5$

$\vec{v}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{v}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Q: $\lim_{n\to\infty} A^n \begin{bmatrix} L(0) \\ R(0) \end{bmatrix} = ?$ probabilities

$L(0)+R(0)=1$

DIAGONALIZATION: $A=BDB^{-1}$, $D=\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$A^n = BD^nB^{-1}$, $D^n = \begin{bmatrix} 1^n & 0 \\ 0 & 0.5^n \end{bmatrix}$, $B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$A^n = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \to A^n = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$

$\frac{1}{2}^n \approx 0$ (negligible)

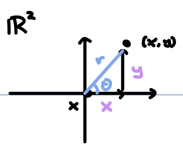
will stabilize over the long term behavior

for larger n, $\begin{bmatrix} L(n) \\ R(n) \end{bmatrix}$ $A^n\begin{bmatrix} L(0) \\ R(0) \end{bmatrix} \approx \begin{bmatrix} 0.6(L(0)+R(0)) \\ 0.4(L(0)+R(0)) \end{bmatrix}$

LONG-TERM behavior: 60% on left, 40% on right

RECALL: POLAR COORDINATES.

$\mathbb{R}^2$



r = dist. from origin
θ = angle w/ + x-axis

POLAR TO RECTANGULAR:

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad x = r\cos\theta$

$(r,\theta) \to (x,y)$ $\quad y = r\sin\theta$

RECTANGULAR TO POLAR:

$g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad r = \sqrt{x^2+y^2}$

$(x,y) \to (r,\theta)$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$g: \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2$

domain of origin

a bit inexact:

1) depends on signs of x & y

2) if $\theta = \pi$, it can also equal $3\pi, 5\pi, 7\pi$, etc.

COMPLEX NUMBERS:

Q: what are eigenvalues/vectors of:

R = rotation by $+90° = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

no real eigenvalues (direction changes during rotation)
or $\lambda = \pm i$, complex-valued eigenvectors

$\det(R - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

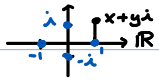
complex #'s = $\{x + yi \mid x \in \mathbb{R},\ i^2 = 1\}$

where

$Re(x+yi) = x$ "real part"

$Im(x+yi) = y$ "imaginary part"

complex plane: $x + yi \rightsquigarrow (x,y)$



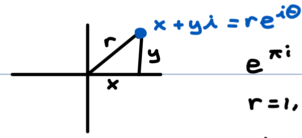
conjugate: $z = x + yi$,
$\bar{z} = x - yi$

addition: $(x+yi) + (a+bi) = (x+a) + (y+b)i$

mult: FOIL $(2+i)(1+3i) = 2 + i + 6i + 3i^2 = -1 + 7i$

division: $\frac{1+3i}{2+i} \cdot \frac{2-i}{2-i} = ?$

complex exponentials: $e^{i\theta} := \cos\theta + i\sin\theta$

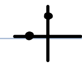


$x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$e^{\pi i} = 1e^{\pi i} = -1$

$r = 1,\ \theta = \pi$

mult: $(5e^{i\frac{\pi}{3}})(4e^{i\frac{2\pi}{3}}) = 20e^{i(\frac{\pi}{3} + \frac{2\pi}{3})} = 20e^{i\pi} = -20$



$e^{i\pi/2} = i$

DIST point to plane:

point = (x_1, y_1, z_1)

coeff. of plane: (a, b, c)

$$\text{distance} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2+b^2+c^2}}$$

→ or: project vector from pt → pt on plane in terms of $\hat{n}$

AREA OF PARALLELOGRAM:

w ▱ v $= |\vec{v} \times \vec{w}| = \sqrt{\langle \text{cross prod. pts} \rangle^2}$

2/24/25 - recitation:

EIGENVECTORS & EIGENVALUES:

Ex: rabbits & foxes represent as $P(n) = \begin{bmatrix} r(n) \\ f(n) \end{bmatrix}$. we know that $p(n+1) = Ap(n)$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

population after n yrs

so $p(1) = Ap(0)$, $P_2 = Ap(1) = A^2 p(0) \to p(n) = A^n p(0)$

in the problem, $p(0) = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$.

calculating $A^n \begin{bmatrix} 30 \\ 20 \end{bmatrix}$ is hard.

on other hand, $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$. in other words, $A\begin{bmatrix} 10 \\ 10 \end{bmatrix} = 2\begin{bmatrix} 10 \\ 10 \end{bmatrix}$. so $A^2\begin{bmatrix} 10 \\ 10 \end{bmatrix} = A\left(2\begin{bmatrix} 10 \\ 10 \end{bmatrix}\right) = 2A\begin{bmatrix} 10 \\ 10 \end{bmatrix} = 2 \cdot 2\begin{bmatrix} 10 \\ 10 \end{bmatrix}$

so $A^n\begin{bmatrix} 10 \\ 10 \end{bmatrix} = 2^n\begin{bmatrix} 10 \\ 10 \end{bmatrix}$.

in other words, $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ is eigenvector of A w/ eigenvalue 2.

could take $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and write $p(0) = 2\begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

HOW did we find eigenvector/eigenvalue?

want $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some λ, i.e. $(A - \lambda Id)\vec{v} = 0$

if $\det(A - \lambda Id) \neq 0$, this system has 1 solution, i.e. $\vec{v} = 0$ ← so λ is not an eigenvalue

if $\det(A - \lambda Id) = 0$, then ∞ solutions ← λ is eigenvalue

each eigenvalue corresponds to eigenvectors

Ex: $\det(A - \lambda Id) = \det\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = (4 - 5\lambda + \lambda^2) - (-2) = 6 - 5\lambda + \lambda^2 = (\lambda-3)(\lambda-2)$. so eigenvalues 2 & 3

Ex: what is eigenvectors w/ eigenvalue 3?

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}}_{\text{subtract } \lambda \text{ from}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $x-2y=0$. so eigenvectors are in form $\begin{bmatrix} 2y \\ y \end{bmatrix}$ for any y.

here, we write P(0) in terms of eigenvectors (diagonalization)

$p(0) = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$. notice $p(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}$. so $A^n p(0) = A^n\left(\begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}\right) = A^n\begin{bmatrix} 10 \\ 10 \end{bmatrix} + A^n\begin{bmatrix} 20 \\ 10 \end{bmatrix} = 2^n\begin{bmatrix} 10 \\ 10 \end{bmatrix} + 3^n\begin{bmatrix} 20 \\ 10 \end{bmatrix}$

- solve for eigenvalues $(A-\lambda Id)\vec{v}=0$
- plug in to find eigenvectors

① find eigenvectors & eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. $\rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)(\lambda^2-1) - 0 + 0 = \lambda^2 - \lambda^3 + \lambda - 1$

$\lambda = 1, \lambda = \pm 1$

$1\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$??

$$\begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix} \rightarrow \begin{bmatrix} .8-\lambda & .2 \\ .3 & .7-\lambda \end{bmatrix} = (.8-\lambda)(.7-\lambda) - (.06) = 0$$

$= .56 - 1.5\lambda + \lambda^2 - 0.06 = 0$

$= .5 - 1.5\lambda + \lambda^2 = 0$?? $\times$

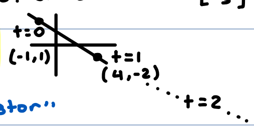
[2/25/25]-lecture #9
- parametrized lines in 2D/3D
- parametrized curves
- tangent vectors

PARAMETRIZED LINES: if we want to "write down" a line in 3D, we need to use "parametrization"

2D: line $3x+5y=2$. "point tester" (plug in pt & see if it satisfies)

pt $(-1,1)$ is on line. direction is $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ $\quad y = -\frac{3}{5}x + 2$ ← -3 y for 5 x

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \underbrace{t}_{\text{parameter "point generator"}}\begin{bmatrix} 5 \\ -3 \end{bmatrix}$



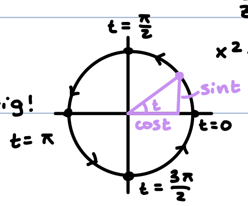
3D: similar, but x, y, z in terms of parameter t.

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \\ 3 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

think of $\vec{r}$ as a map: $\mathbb{R} \rightarrow \mathbb{R}^3$, $t \rightarrow (x,y,z)$

we can write as
$x(t) = -2-5t$
$y(t) = -7+3t$
$z(t) = 3+2t$

EX of PARAMTRIZATION: trig!



$x^2+y^2=1$ (pt. tester)

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

trace once: $0 \le t \le 2\pi$ ← specify bounds

*can have multiple parametrizations of same curve!

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}$

POLAR EQNS: $r = f(\theta)$,

$r = \theta \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} \theta\cos\theta \\ \theta\sin\theta \end{bmatrix} = \vec{r}(\theta)$ $\quad \vec{r} = |\vec{r}|$

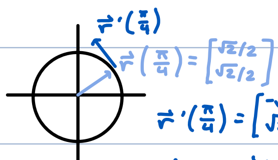
Ex: helix!

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$

TANGENT VECTORS:

ex: $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

differentiate: $\vec{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$



$\vec{r}'(\frac{\pi}{4}) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

$\underbrace{\vec{r}'(t)}_{\text{velocity}} \perp \vec{r}(t)$! ONLY when circle

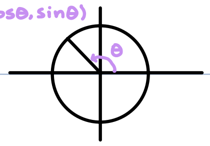
speed $= \underbrace{|\vec{r}'(t)|}_{\text{magnitude}} = v(t)$

here $v(t) = |\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

[2/26/25] - recitation - test review

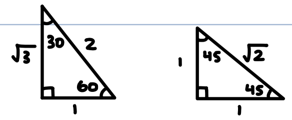
UNIT CIRCLE & TRIG:

$(\cos\theta, \sin\theta)$



$\cos\theta = \cos -\theta$

$\sin\theta = \sin -\theta$



LINEAR TRANSFORMATION:

Suppose i have linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

such that $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-8\\7\end{bmatrix}$

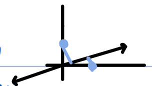
→ corresponding matrix $M = \begin{bmatrix}3 & -8\\5 & 7\end{bmatrix}$

then $T\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3 & -8\\5 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3a-8b\\5a+7b\end{bmatrix}$

check: using this formula $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3-0\\5+0\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$

$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0-8\\0+7\end{bmatrix} = \begin{bmatrix}-8\\7\end{bmatrix}$

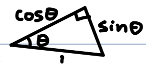
EX: find matrix for T that projects orthogonally onto the line $y = \frac{x}{2}$



STEP 1: what is $T\begin{bmatrix}1\\0\end{bmatrix}$ & $T\begin{bmatrix}0\\1\end{bmatrix}$

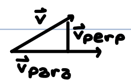
- approach 1: find angle b/t line & x-axis & use cosine

→ $\begin{bmatrix}1\\0\end{bmatrix}$ should go to $\cos\theta \cdot \hat{u}$ in direction of the line



PROJECTIONS & COMPONENTS:

in other words, this problem of finding vector projection of $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ along line spanned by $\vec{w}: \begin{bmatrix}2\\1\end{bmatrix}$



$\vec{v}\cdot\vec{w} = \vec{v}_{para}\cdot\vec{w} = |\vec{v}_{para}|\cdot|\vec{w}|\cdot 0 \to |\vec{v}_{para}| = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} \to \vec{v}_{para} = |\vec{v}_{par}|\cdot\frac{\vec{w}}{|\vec{w}|}$

for $\begin{bmatrix}1\\0\end{bmatrix}$: component along line $= \frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}}$

vector component is $\frac{2}{\sqrt{5}} \cdot \frac{\begin{bmatrix}2\\1\end{bmatrix}}{\sqrt{5}} = \begin{bmatrix}4/5\\2/5\end{bmatrix}$ → so $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4/5\\2/5\end{bmatrix}$

SPAN: the span of set of vectors is the set of linear combos of them.

→ in $\mathbb{R}^3$: · one vector $\vec{v}$ spans line {all multiples of $\vec{v}$}

· two vect. v_1 & v_2 span a plane if they are linearly independent (ex: $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. span is set of points $\begin{bmatrix}a\\b\\0\end{bmatrix}$

- if they are dependent, they span a line (ex: $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ & $v_2 = \begin{bmatrix}2\\0\\0\end{bmatrix}$, the span is pts $\begin{bmatrix}a\\0\\0\end{bmatrix}$

· three vect: lin. ind → span a volume, all of $\mathbb{R}^3$

AKA det of matrix w/ cols is nonzero ("volume")

EX: $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ span a plane but are not linearly independent b/c $v_1 + v_2 - v_3 = 0$

the # of elements in a basis of $\mathbb{R}^n$ is always n.

if set of vectors spans $\mathbb{R}^n$, we must have $\geq n$ vectors, and if it has n vectors, these form a basis.

if set of vectors in $\mathbb{R}^n$ is linearly indep, we must have $\leq n$ vectors, and if $= n$ vectors, it forms a basis.

given 3 vectors, set forms a basis ⟺ spans ⟺ linearly independent ⟺ $\det \neq 0$

PROPERTIES OF det:

- "area-transformation factor" of the corresponding linear transformation.
- tests whether columns of matrix form a basis/span/linearly independent
- tests whether matrix has an inverse
- tests whether $A\vec{x} = \vec{b}$ has one solution or not (when A has inverse, $\vec{x} = A^{-1}\vec{b}$)
  - if $\det = 0$, there's a redundancy, so either no solutions or ∞ solutions

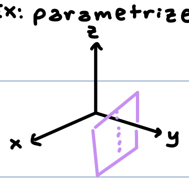
$\begin{cases} x+y=3 \\ x+y=4 \end{cases}$ // (parallel)

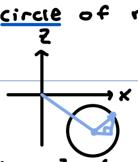
$\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$ / (same line)

2/29/25 - lec

PARAMETRIZATION EXS:

Ex: parametrize circle of rad 3, centered @ (5, 7, -9) & parallel to xz-plane.





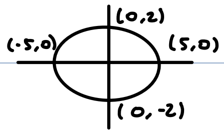
$(x-5)^2 + (z-9)^2 = 3^2$

$\begin{bmatrix}x\\z\end{bmatrix} = \begin{bmatrix}5\\-9\end{bmatrix} + 3\begin{bmatrix}\cos t\\\sin t\end{bmatrix}$

$\boxed{\begin{aligned} x &= 5 + 3\cos t \\ y &= 7, \ 0 \leq t \leq 2\pi \\ z &= -9 + 3\sin t \end{aligned}}$

Ex: ellipse → stretch cost, sint by diff. amounts

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5\cos t \\ 2\sin t \end{bmatrix}$$



$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

PARAMETRIC MOTION: $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ traces out a path in space over time t

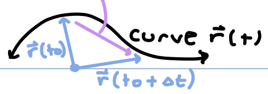
$\vec{r}(t)$ = pos.

$\vec{r}\,'(t)$ = vel.

$\vec{r}\,''(t)$ = accel.

Why can't we simply differentiate $\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$ to get velocity?

$\vec{r}(t_0+\Delta t) - \vec{r}(t_0) = \Delta\vec{r}$



∞ tiny piece of curve

avg. rate of change pos. $\frac{\Delta\vec{r}}{\Delta t}$

instantaneous $= \lim_{\Delta t\to 0} \frac{\vec{r}(t+t_0) - \vec{r}(t_0)}{\Delta t} = \vec{r}(t_0)$ — tangent, similar to secant line → tan line

WRITE $\frac{d\vec{r}}{dt} = \vec{r}(t)$, or $d\vec{r} = \vec{r}\,'(t)\,dt$

$\Delta\vec{r} \approx \vec{r}\,'(t)\,\Delta t$

$|\Delta\vec{r}|$ approx. length of curve from t to $t+\Delta t$

think of $|d\vec{r}|$ as length of tiny part of curve

$\sum_{i=1}^{n} |\Delta\vec{r}_i|$ approximates arc len.

$$\text{arc len} = \int_{t=a}^{t=b} |d\vec{r}| = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$|\vec{r}\,'(t)| = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

$$\vec{r}(t) \underset{\int dt}{\overset{\frac{d}{dt}}{\rightleftarrows}} \vec{v}(t) \underset{\int dt}{\overset{\frac{d}{dt}}{\rightleftarrows}} a(t)$$

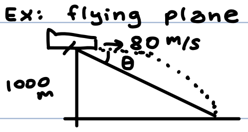
EX: pt moving in xy-plane

<u>KNOW</u>: @ $t=0$, $\vec{r}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{v}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \hat{\imath} - \hat{\jmath}$ & $\vec{a}(t) = \begin{bmatrix} -2 \\ 6t \end{bmatrix}$

<u>FIND</u>: $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \begin{bmatrix} -2 \\ 6t \end{bmatrix} = \begin{bmatrix} -2t \\ 3t^2 \end{bmatrix} + \vec{c}$

<u>PLUG</u>: $t=0$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \vec{c}$ $= \begin{bmatrix} -2t+1 \\ 3t^2-1 \end{bmatrix}$

Ex: flying plane horizontally @ 1000m, 80 m/s. want to drop on specific pt. what angle?



$\vec{a}(t) = -9.8\,\hat{k}$

$\vec{v}(0) = 80\,\hat{\imath}$

$\vec{r}(0) = 1000\,\hat{k}$

$\vec{v}(t) = 80\,\hat{\imath} - 9.8t\,\hat{k}$

$\vec{r}(t) = 80t\,\hat{\imath} - 4.9t^2\,\hat{k} + 1000\,\hat{k}$

want height = 0

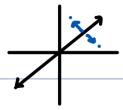
$1000 - 4.9t^2 = 0$

$t = \sqrt{\frac{1000}{4.9}}$

hori. dist $= 80\sqrt{\frac{1000}{4.9}}$

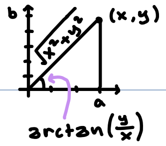
$\theta = \tan^{-1}\left(\frac{1000}{80\sqrt{\frac{1000}{4.9}}}\right)$

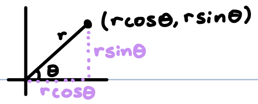
[3/3/25] - rec - 1st rec after midterm 1

test q: reflect across line: 

SNEAK PREVIEW: $\int_{-\infty}^{\infty} e^{-x^2}dx$ using polar coords is easier!

RECTANGULAR VS. POLAR COORDS in $\mathbb{R}^2$:

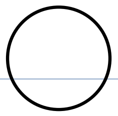




(r,θ) polar $\to$ $(r\cos\theta, r\sin\theta)$ rectangular

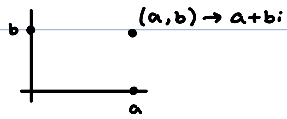
(x,y) rect. $\to$ $\left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$ polar

COMPLEX NUMBERS:



$\pi = 3.1415\ldots$

add $\sqrt{-1} = i$ to real #'s. so we should also add $a+bi$ for a,b real #'s.



if $a+bi$ corresponds to rectangular coords, what do polar coords correspond to?

$e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \cdots$$
$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{6} + \frac{\theta^4}{24} + \cdots$$

So, real part is $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \ldots$ (cosθ power series)

imaginary part $i\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{5!} - \ldots\right)$ (sinθ power series)

they correspond to $re^{i\theta}$

ADD/SUBTRACT are much easier in rectangular coords: $(a+bi)+(c+di) = (a+c) + (b+d)i$

MULTIPLICATION/DIVISION much easier in polar coords: $r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

- we can still multiply/divide in rect. coords: $(a+bi)(c+di) = ac + bci + adi + bdi^2 = (ac-bd) + i(bc+ad)$

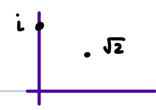
$$\frac{a+bi}{c+di} \cdot \underbrace{\frac{c-di}{c-di}}_{\text{conjugate}} = \frac{(a+bi)(c-di)}{c^2-d^2i^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

EX: compute $\sqrt{i}$ using polar coords or rect.

→ POLAR: find r,θ s.t. $(re^{i\theta}) = i = e^{i\pi/2}$

we know $(re^{i\theta})^2 = r^2e^{i2\theta}$ so we know $r^2 = 1$ & $2\theta = \frac{\pi}{2}$

$\leadsto r = 1$ $\qquad \leadsto \theta = \pi/4$ or $\frac{\pi}{4} + \pi$



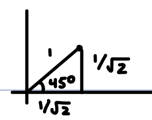
→ RECT: find (a,b) s.t. $(a+bi)^2 = i$

we know $(a+bi)^2 = a^2 + 2abi - b^2$. so $a^2 - b^2 = 0$ & $2abi = i$

$a = \pm b$ $\qquad 2a^2 = 1$

$a = \pm\frac{1}{\sqrt{2}}$

so we get $\pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$



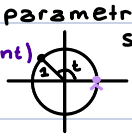
PARAMETRIZED CURVES: given some function $p(t) = (x(t), y(t), z(t))$ describing the position of some object as func. of time.

EX: parametrize line b/t (1,2,3) & (4,4,4). Say we're at (1,2,3) @ time $t=0$ & (4,4,4) @ $t=1$.

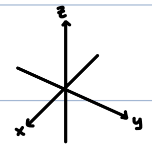
$x(t) = 1+3t$, $y(t) = 2+2t$, $z(t) = 3+t$ (linear, correct @ $t=0$ & $t=1$)

EX: parametrize going CCW around circle of radius 1 @ constant speed (1 unit every 2π sec) starting @ (1,0)

$p(t) = (\cos t, \sin t)$



[3/4/25] - lecture 11 — { · surfaces $z=f(x,y)$ · partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ · linear approx.



FUNCTIONS: $f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to z = f(x,y)$

GRAPH of $z=f(x,y)$ is a surface

Ex: $z = x^2+y^2$ (bowl)

How to visualize?

① table of values

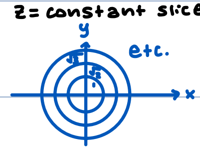
0	4	1	0
-1	5	2	1
-2	8	5	4
y/x	-2	-1	0

② look @ x = constant, vertical; y = constant, slices

$y=-2 \to z = x^2+4$
$y=-1 \to z = x^2+1$
$y=0 \to z = x^2$
$y=1 \to z = x^2+1$
$y=2 \to z = x^2+4$

all parabolas (x = const. also parabolas)

③ "level curves/contours"
z = constant slices



$z=1: x^2+y^2=1$
$z=2: x^2+y^2=2 \to \text{rad} = \sqrt{2}$
$z=3: x^2+y^2=3 \to \text{rad} = \sqrt{3}$

* "contour plot"
* "topographical map"

④ ask computer to 3D graph

Ex: $z = x^2 - y^2$ (saddle)

y-const: upward-facing parabolas
x-const: downward-facing para.
contours: $x^2-y^2=c$

$z=0: x^2-y^2=0 \to x^2=y^2$
$y = \pm x$

$z=1: x^2-y^2=1$ $\quad y^2 = x^2-1$
$y = \pm\sqrt{x^2-1}$

PARTIAL DERIVATIVES:

partial-x: pretend y is constant, diff. w/ x

NOTATION: $z=f(x,y)$. $\frac{\partial z}{\partial x}$, f_x, z_x

Ex: $z = x^2+y^2 \to \frac{\partial x}{\partial z} = 2x+0$

partial-y: hold x constant

Ex: $z = x^3y+2x$

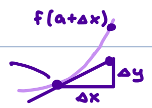
$\frac{\partial z}{\partial x} = 3x^2y+2$

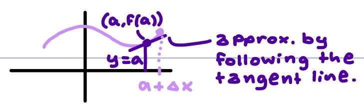
$\frac{\partial z}{\partial y} = x^3$

*review chain rule, num/denom, etc

LINEAR APPROX:

recall: $y=f(x)$. @ pt $(a, f(a))$





slope $= f'(a) = \frac{\Delta y}{\Delta x}$

$\Delta y = f'(a)\cdot\Delta x$

$f(a+\Delta x) - f(a) \approx f'(a)\cdot\Delta x$

$\frac{dy}{dx}\cdot\Delta x$

$f(a+\Delta x) \approx f(a) + \frac{dy}{dx}\cdot\Delta x$

now $z=f(x,y)$ near $(a, b, f(a,b))$

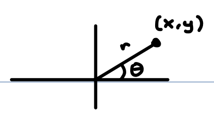
approximate $f(a+\Delta x, b+\Delta y)$

if $\Delta y=0$, $f(a+\Delta y, b) \approx \frac{\partial f}{\partial x}\cdot\Delta x$

if $\Delta x = 0$, $f(a, b+\Delta y) \approx \frac{\partial f}{\partial y}\cdot\Delta y$

TOTAL CHANGE $\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$

[3/5/25] - office hours

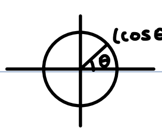


a pt in $\mathbb{R}^2 \setminus 0$ can be uniquely described as (r, θ) where $r>0$ & $0 \le \theta \le 2\pi$

or un-uniquely described by (r,θ) whee $r>0$ & $-\infty < \theta < \infty$

($\theta + k\cdot 2\pi$ is same as θ)

COMPLEX NUMBER: uniquely described by $a+bi$, where a & b are real #s.



$(\cos\theta, \sin\theta) \longleftrightarrow \cos\theta + i\sin\theta = e^{i\theta}$

a general nonzero complex # can be described as $re^{i\theta}$
($r>0$, θ well-def up to multiples of 2π)

$r_1e^{\theta_1}\cdot r_2e^{\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

Ex: find roots of $z^2=-8i$ $= \cos\theta + i\sin\theta \to \theta = -\frac{\pi}{2}$

nth degree poly always has n roots $\quad r=8$

$\therefore z^2$ has 2 roots

$8e^{-i\pi/2} =$

$re^{i\theta} = se^{iw} \to r=s$ & $\theta = w+$ some multiple of 2π

$z = re^{i\theta} \to z^2 = r^2e^{2i\theta} = 8e^{-i\pi/2}$

$r^2=8 \to r=\sqrt{8}$

$2\theta = -\frac{\pi}{2} + \text{multiples of } 2\pi$

$\theta = -\frac{\pi}{4} + \text{multiples of } \pi$

$z = \sqrt{8}\, e^{i(-\frac{\pi}{4} + \pi k)}$

$= \sqrt{8}\, e^{-\frac{\pi}{4} i}$ or $\sqrt{8}\, e^{(-\frac{\pi}{4}+\pi)i}$ ← only 2 solutions! $\sqrt{8}\, e^{-\pi/4\, i}$ & $\sqrt{8}\, e^{-5\pi/4\, i}$

[recitation] - **PARAMETRIC EQNS:** used to describe $p(t) = (x(t), y(t), z(t))$ the pos. of particle as funct. of t.

what's velocity? $p'(t) = (x'(t), y'(t), z'(t))$

speed? $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

direction? $p'(t)/s(t)$ ← unit vect. in direction of v

acceleration? $p''(t) = (x''(t), y''(t), z''(t))$

how far does particle travel over some interval of time? → $\int_a^b \text{speed}\, dt$ (integral of speed over integral of time is distance travelled in that time)

Ex:

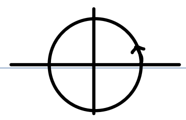
1) motion along line @ constant speed: $p(t) = \underbrace{(x_0, y_0, z_0)}_{\text{pos. @ } t=0} + t\underbrace{(a,b,c)}_{\text{direction of line}}$

2) motion with constant acceleration: given $p''(t) = (a,b,c)$ we know $p'(t) = (a_0 + ta, b_0 + tb, c_0 + tc)$ for some (a_0, b_0, c_0) determined by $p'(0)$

and $p(t) = (A_0 + ta_0 + \frac{t^2}{2}a,\ B_0 + tb_0 + \frac{t^2}{2}b,\ C_0 + tc_0 + \frac{t^2}{2}c)$

for some A_0, B_0, C_0 determined by $p(0)$

3) motion around a circle radius r @ constant speed CCW starting @ $r=3$



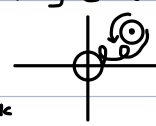
$p(t) = (r\cos t, r\sin t)$

<u>variations:</u> if center of circle @ (a,b), then $p(t) = (a + \cos t, b + \sin t)$

- if CW, then $p(t) = (r\cos(-t), r\sin(-t))$
- if want to rotate faster, $p(t) = (r\cos(17t), r\sin(17t))$ ← 17x faster
- if want to start somewhere other than 3 o'clock, $p(t) = (r\cos(t+\theta), r\sin(t+\theta))$
- if we know 1 rev takes time x & use $(\cos(at), \sin(at))$, how long does 1 rev. take?

rad = a try $a = \frac{2\pi}{x}$ → at goes from 0 to 2π; so $\frac{2\pi}{a} = x$

<u>PROBLEM:</u> hockey puck sliding across ice w/ constant velocity $(1,1)$ m/s starting @ $(0,0)$.
it's rotating CCW @ speed 2 rev/sec
find eqn. of pos. of pt on boundary that started @ $(0,a)$.



we'll start by finding an eqn. $c(t)$ for center of hockey puck

then $p(t) = \underbrace{c(t)}_{\text{pos. of center}} + \underbrace{\vec{CP}(t)}_{\text{pos. of pt. relative to center}}$

$c(t) = (t, t) = t(1,1)$

$\vec{CP}(t) = (a\cos(4\pi t), a\sin(4\pi t))$

2 rev = 1 sec $x = \frac{1}{2}$

1 rev = $\frac{1}{2}$ sec $a = \frac{2\pi}{1/2} = 4\pi$

[3/5/25] - lecture - for surfaces $z = f(x,y)$:
- tangent planes
- directional derivatives
- the (2D) gradient

<u>TODAY:</u> focus on $z = f(x,y)$ <u>MORE GENERAL:</u> $g(x,y,z) = \text{constant}$

TANGENT PLANES: most general plane eqn is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, pt $P = (x_0, y_0, z_0)$, normal vect. $a\hat{i} + b\hat{j} + c\hat{k}$

if plane ≠ vertical, then $c \neq 0$. so rewrite as $z - z_0 = m(x-x_0) + n(y-y_0)$

$\Delta z = m\Delta x + n\Delta y$

m = "x slope": if $\Delta y = 0$, $m = \frac{\Delta z}{\Delta x}$, slope in x direction

n = "y slope" similarly

for $z = f(x,y)$, @ pt (a,b) zoom in @ P and surface "flattens out", see tangent plane

"x slope" of tangent plane = instantaneous R.O.C. in positive x-direction, holding y constant = $f_x(P)$

similarly, y-slope $f_y(P)$ =

$\Delta z = (f_x(P))\Delta x + (f_y(P))\Delta y$ → $\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ ← linear approx.

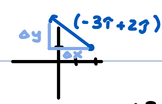
plug these in, $z - f(a,b) = \underbrace{(f_x(a,b))}_{\#}(x-a) + \underbrace{(f_y(a,b))}_{\#}(y-b)$ ← tangent plane eqn

$0 = f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0)$

DIRECTIONAL DERIVATIVE:

<u>Ex:</u> $f(x,y) = x^2y + x$, pt $(2,1)$

<u>Q:</u> what is instantaneous ROC in direction $\underbrace{-3}_{\Delta x}\hat{x} + \underbrace{2}_{\Delta y}\hat{y}$



to figure this out, follow tangent plane & compute $\frac{\text{rise}}{\text{run}} = \frac{\Delta z}{\Delta \text{horizontal}} = \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{f_x\Delta x + f_y\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{5(-3) + 4(2)}{\sqrt{(-3)^2 + 2^2}} = \frac{-7}{\sqrt{13}}$

could rewrite $\frac{f_x\Delta x + f_y\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = f_x\left(\frac{\Delta x}{\sqrt{\cdots}}\right) + f_y\left(\frac{\Delta y}{\sqrt{\cdots}}\right)$

$f(x,y) = x^2y + x$ $f_x = 2xy + 1$ $f_y = x^2$

$f_x(2,1) = 5$ $f_y(2,1) = 4$

factor as dot product: $\langle f_x, f_y\rangle \cdot \langle u_1, u_2\rangle$

where $\hat{u} = \langle u_1, u_2\rangle = \left\langle \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}, \frac{\Delta y}{\sqrt{\cdots}}\right\rangle$

the vector $\langle f_x, f_y \rangle$ is called the gradient of f, written $\vec{\nabla} f$ or $\overrightarrow{\text{grad}}\, f$

DEF: given $z = f(x,y)$ @ pt. P, directional derivative of f @ P in direction $\hat{u}$ = inst. ROC of f @ P in direction $\hat{u}$.
written $f_{\hat{u}}(P)$

THM: $f_{\hat{u}}(P) = \big((\vec{\nabla} f)(P)\big) \cdot \hat{u}$

Ex: $(5\hat{i} + 4\hat{j}) \cdot \left(\frac{-3}{\sqrt{3}}\hat{i} + \frac{2}{\sqrt{13}}\hat{j}\right) = \frac{-7}{\sqrt{13}}$

GEOMETRIC INTERPRETATION OF THE GRADIENT:

$f_{\hat{u}}(P) = \big((\vec{\nabla} f)(P)\big) \cdot \hat{u} = |(\vec{\nabla} f)(P)| \cdot \cancel{|\hat{u}|}^{1} \cdot \cos\theta$, where θ = angle b/t $\hat{u}$ & $\overrightarrow{\text{grad}}\, f$

NOTE: $-1 \le \cos\theta \le 1 \rightarrow -|(\vec{\nabla} f)(P)| \le f_{\hat{u}}(P) \le |(\vec{\nabla} f)(P)|$

EX: $f(x,y) = x^2 y + x$, $P = (2,1)$

$\hat{u} = \left(\frac{5}{\sqrt{41}}, \frac{4}{\sqrt{41}}\right)$, same direction as grad f, $f_{\hat{u}}(P) = \langle 5,4 \rangle \cdot \left\langle \frac{5}{\sqrt{41}}, \frac{4}{\sqrt{41}} \right\rangle = \frac{41}{\sqrt{41}} = \sqrt{41}$ & $|\langle 5,4 \rangle| = \sqrt{41}$

THM: $(\vec{\nabla} f)(P)$ is a vector:
- magnitude = max directional derivative of f @ P
- direction = direction of "steepest ascent"

[3/6/25 - lecture] →
- gradient as vect. field
- tan lines to level curves
- level surfaces, tan planes to level surf.
- total deriv (+ matrix)

for $z = f(x,y)$, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ $\quad (\vec{\nabla} f)(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ $\quad \vec{\nabla} f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

(draw a vector coming out of each pt, get a pic. of vector field)

RECALL: geometric interp of grad for $z = f(x,y)$

DIRECTION: of max directional deriv.
- $\perp$ to level curves $\quad ((f_{\hat{u}})(P) = (\vec{\nabla} f)(P) \cdot \hat{u} = |\vec{\nabla} f| \cos\theta$
- if $\theta = 90°$, $\perp$ to grad, then $f_{\hat{u}} = 0$ ← following level curve

MAG: max directional deriv.

APPLICATION (to finding tan lines):

Ex: follow $x^3 + y^3 = 9xy$ ← (4,2) lines on this

Q: Eqn of tan. line @ this point?

consider $f(x,y) = x^3 + y^3 - 9xy$ ← folium is $z=0$ level curve

(can do this for any curve in xy-plane)

then $(\vec{\nabla} f)(P)$ is $\perp$ to level case

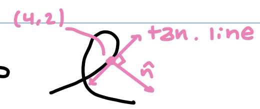
$\vec{\nabla} f = \langle 3x^2 - 9, 3y^2 - 9x \rangle$

$(\vec{\nabla} f)(4,2) = \langle 30, -24 \rangle = 6\langle 5, -4 \rangle$

so $\hat{n} = \langle 5, -4 \rangle$ is $\perp$ to level curve / folium

line through (4,2) w/ normal vect. $\langle 5,-4 \rangle$ is $5(x-4) - 4(y-2) = 0$

$\boxed{5x - 4y = 12}$



GO UP DIMENSION: $w = f(x,y,z)$ $\quad f: \mathbb{R}^3 \rightarrow \mathbb{R}$ $\quad \vec{\nabla} f = \langle f_x, f_y, f_z \rangle$ ← can draw 3D vect. field

- same geometric interp. & same direct derivative formula $f_{\hat{u}} = (\vec{\nabla} f) \cdot \hat{u}$

SET $f(x,y,z) = c$, get "level surfaces"

- $(\vec{\nabla} f)(P)$ is $\perp$ to level surface through P.

Ex: $w = f(x,y,z) = x^2 + y^2 + z^2$

level surfaces are $x^2 + y^2 + z^2 = c$, sphere of radius $\sqrt{c}$

$(\vec{\nabla} f) = \langle 2x, 2y, 2z \rangle = 2\langle x,y,z \rangle$ ← point away from origin $\perp$ to sphere

Ex: $w = f(x,y,z) = x^2 + y^2 - z^2$

LEVEL SURF: $w = 0 \rightarrow x^2 + y^2 - z^2 = 0 \rightarrow z^2 = x^2 + y^2$ (double cone)

$w = 4 \rightarrow x^2 + y^2 = z^2 + 4$ horizontal cross sections are circles (one-sheet hyperboloid)

$w = -4 \rightarrow x^2 + y^2 = z^2 - 4$ nothing b/t $-2 < z < 2$ (two-sheeted hyperboloid)

TAN. PLANE: $x^2 + y^2 - z^2 = 6$ @ (1,3,2).

$\vec{\nabla} f = \langle 2x, 2y, -2z \rangle$

$\hat{n} = (\vec{\nabla} f)(P) = \langle 2, 6, -4 \rangle$

plane: $2(x-1) + 6(x-3) - 2(z-2)$

[3/10/25] - rec

APPLICATIONS OF PARTIAL DERIVATIVES:

$$\frac{\partial f(x,y,\dots)}{\partial x} = f_x$$

Slogan: every function looks linear on a small enough scale
(derivative tells you the best linear approx. to a func. @ particular pt)

Ex: $f(x,y) = 3\sin(x)y + \frac{(\cos x)^3}{y}$

what's best linear approx. for f near $(x,y)=(0,1)$

suppose $f(x,y) = \overbrace{ax+by+c}^{g(x)}$ is best linear approx. then the work below shows best linear approx is $f(x,y) \approx 3x+2y+2$ near $(0,1)$

we want $f(0,1)=1$ should equal $a(0)+b(1)+c = b+c$

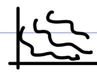
$$\frac{\partial f}{\partial x}(0,1) = 3\cos(x)y + \frac{-3(\cos x)^2 \sin x}{y} \Rightarrow \frac{\partial f}{\partial x}(0,1) = 3 = \frac{\partial(ax+by+c)}{\partial x} = a$$

$$\frac{\partial f}{\partial y} = 3\sin x - \frac{(\cos x)^3}{y^2} \Rightarrow \frac{\partial f}{\partial y}(0,1) = -1 \qquad \frac{\partial(ax+by+c)}{\partial y} = b \Rightarrow c = 2$$

APPLICATIONS:

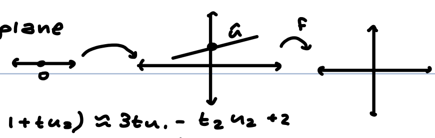
1) approximating functions near known values: in our previous example, if we want to know $f(1.1,2)$ we can use linear approx $f(1.1,-2) \approx 3(.1) - (1.2) + 2 = 1.1$

2) tangent lines to level curves: in our example, to find tan. line to level curve through $(0,1)$, aka level curve $f(x,y)=1$, at pt $(0,1)$ pretend $f(x,y) = 3x-y+2$, and level curve is a set of pts $3x-y+1=0 \Leftrightarrow y=3x+1$



3) tan. plane to $z=f(x,y)$. in our q to find tan. plane @ pt $(0,1,1)$, we pretend surface is $z=3x-y+2$ & this is the tan. plane

4) directional derivatives: $f_{\hat{u}}(0,1) = \frac{d}{dt}\Big|_{t=0} f(p(t))$

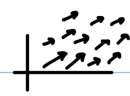


where $p(t) = (0,1) + t\hat{u}$. ex: $\hat{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

as usual, pretend $f(x,y) \approx 3x-y+2$. then $f(p(t)) = f(tu_1, 1+tu_2) \approx 3tu_1 - tu_2 + 2$

$\frac{d}{dt} f(\hat{u}t) \approx 3u_1 - u_2$. in other words, $f_{\hat{u}}(0,1) = \hat{u} \cdot \left(\frac{\partial f}{\partial x}(0,1), \frac{\partial f}{\partial y}(0,1)\right)$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \nabla \vec{f}$ "gradient"



& we care about this b/c $f_{\hat{u}}(\vec{p}) = \hat{u} \cdot \nabla f(p)$

the gradient points in direction where f increases the fastest b/c $\hat{u} \cdot \nabla f(p) = 1 \cdot |\nabla f(p)|$ is maximized when $\cos\theta = 1$ $\nabla f(p)$ & $\hat{u}$ point in same direction

Ex: suppose P is local maximum. what is $\nabla f(p)$? $\nabla f(P) = 0$ b/c @ the peak - can't go up more

[3/11/25] - lecture 14 →
- total derivative
- differentials
- multivar. chain rule
- (if time) higher order partials

REMINDER: when $z = f(x,y)$, all directional derivatives are determined by $\overrightarrow{\text{grad}} f$, using formula $D_{\hat{u}} f = (\vec{\nabla} f) \cdot \hat{u}$

ASIDE: matrix transpose: given $m \times n$ matrix A, A^T is the $n \times m$ matrix found by changing rows into columns

EX: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}}_{\text{"column vector"}}^T = \underbrace{[4,5,6,7]}_{\text{"row vector"}}$ $\quad a^T_{ij} = a_{ji}$

GIVEN 2 column vectors $\vec{v}$ & $\vec{w}$, ex $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, dot product $= \overbrace{[1\ 2\ 3]\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}^{\text{matrix multiplication}}$, we can write $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

we think of gradient as row vector = 1×2 matrix $\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right]$

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \left[\frac{\partial z}{\partial x}\ \frac{\partial z}{\partial y}\right]\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

then $(D_{\hat{u}} f)(P)$:

$z = f(x,y) = x^2 - y^2$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^1$

$\vec{\nabla} f: [2x, -2y]$

$\vec{\nabla} f(3,1) = [6, -2]$

$\hat{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$, $(D_{\hat{u}} f)(P) = \overbrace{[6,-2]\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}}^{\text{matrix multiplication}} = 3 - \sqrt{3} \approx \Delta z$

then $(D_{\hat{u}} f) = (\underbrace{(\vec{\nabla} f)(P)}_{1\times 2})\underbrace{\hat{u}}_{2\times 1}$

matrix

grad f = "total derivative of f" = (Df)

$(D_{\hat{u}} f)(P) = ((Df)(P))\hat{u}$

TOTAL DERIVATIVE: given $f: \mathbb{R}^{n} \to \mathbb{R}^{m}$, @ pt. $P \in \mathbb{R}^n$, the total derivative of f @ P is $m \times n$ matrix $(Df)(P)$ = "best linear approx. of f near P"
1x2

Ex: $z = x^2+y^2 = f_1(x,y)$, $w = xy = f_2(x,y)$ $\quad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2+y^2 \\ xy \end{bmatrix}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^2$

then $Df = \underbrace{\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}}_{2\times 2 \text{ matrix}}$ = "Jacobian"

Ex: $\begin{bmatrix} \Delta z \\ \Delta w \end{bmatrix} \approx \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$ *useful for vector fields, switch var.

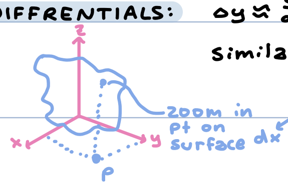
$\begin{bmatrix} dz \\ dw \end{bmatrix} = \begin{bmatrix} z_x & z_y \\ w_x & w_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$

Ex: $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ $\quad \vec{r}: \mathbb{R}^1 \to \mathbb{R}^3$

$(D\vec{r})(t) = \vec{r}'(t) = \underbrace{\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}}_{3\times 1 \text{ matrix}}$

DIFFRENTIALS: $\Delta y \approx \frac{dy}{dx}\Delta x \xrightarrow{\Delta x \to 0} dy = \frac{dy}{dx}dx$

similarly, $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \xrightarrow[\Delta y \to 0]{\Delta x \to 0} dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$



ex: $dz = 6dx - 2dy$
an eqn. for tangent plane!
(through new (0,0,0))

MULTIVARIABLE CHAIN RULE:

$\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^1 \xrightarrow{g} \mathbb{R}^1$, x_0, $y_0 = f(x_0)$, $z_0 = g(y_0)$, $g \circ f$

$(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0)$

$\mathbb{R}^1 \xrightarrow{f'(x_0)} \mathbb{R}^1 \xrightarrow{g'(y_0)} \mathbb{R}^1$, $(g\circ f)'(x_0) = c$ $\qquad c = b \cdot a$ $\qquad \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

a, b, c num

$z = f(x,y) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{r}(t)$

$\mathbb{R}^1 \xrightarrow{\vec{r}} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^1$

$t_0 \longrightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \longrightarrow z_0 = f\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right)$

$\frac{dz}{dt} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$

$D(f \circ \vec{r}) = (Df)(D\vec{r})$

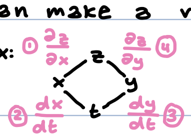
use $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$

divide Δt: $\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}$

$\Delta t \to 0$: $\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

multivariable chain rule
(for 2 variables)
instead of products of nums, we have products of matrices

*can make a variable dependency chart

Ex:


"multiply along paths then add"

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
① ② ③ ④

[3/12/25] - office hours

Ex: cylindrical tank volume $\pi r^2 h$ & inflating such that r incr. by 2 cm/s, h incr. by 5 cm/sec.
what is volume @ $r = 1.02$ & $h = 4.98$?

$V(r,h)$

$\frac{\partial V}{\partial r} = 2\pi r h$ $\qquad \frac{\partial V}{\partial h} = \pi r^2$ $\qquad$ also $V(1,5) = 5\pi$

$V(1+\Delta r, 5+\Delta h) \approx 2\pi r h \Delta r + \pi r^2 \Delta h + 5\pi$
$\Delta r = +0.02$ $\quad \Delta h = -0.02$

$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$f = f(x,y,z) \rightsquigarrow Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

EX: COMPLEX #S

TYPES: · solve $\frac{1+2i}{3+4i} = \frac{(1+2i)}{(3+4i)} \cdot \frac{(3-4i)}{(3-4i)} = \frac{3-4i+6i+8}{9+16} = \frac{11}{25} + \frac{2i}{25}$

· find all the roots of ...

$a = r \quad b = \theta$

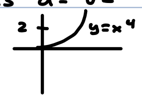
EX: find 4th roots of $2i$: if ae^{ib} is 4th root of $2i$, then $(ae^{ib})^4 = \boxed{a^4 e^{4ib} = 2e^{i\pi/2}}$

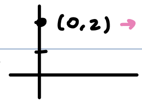
$r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ exactly when $r_1 = r_2$ and $\theta_1 = \theta_2$ + some multiple of 2π

so $a^4 = 2$ and since a is ⊕ and real #, this means $a = \sqrt[4]{2}$

$4b = \frac{\pi}{2} + k \cdot 2\pi$ for some k

$\boxed{b = \frac{\pi}{8} + k \cdot \frac{\pi}{2}}$ ← 4 unique solutions



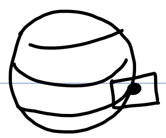


$(0,2) \rightarrow r=2, \theta = \pi/2$

$(x,y) \rightsquigarrow r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

EX: tangent plane @ a 3D point



$z = f(x,y), \quad P = (1,2,3)$

$z = f_x(x-1) + f_y(y-2) + f_z(z-3)$

$z = 5x + 8y - 18 = f(x,y)$ $\qquad g(x,y,z) = 0 = 5x + 8y - 18 - z$

$\Rightarrow \frac{\partial g}{\partial x}(1,2,3)\cdot(x-1) + \frac{\partial g}{\partial y}(1,2,3)\cdot(y-2) + \frac{\partial g}{\partial z}(1,2,3)\cdot(z-3) = 0$

$= 5(x-1) + 8(y-2) - 1(z-3) = 0$

$\vec{n} = \langle 5, 8, -1 \rangle$

EX: DIRECTIONAL DERIVATIVE

[3/12/25]-rec

$\mathbb{R}^3 \to \mathbb{R}^1$

$f(x,y,z) \to \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

recall: given some direction $\hat{u}$ & pt. P, consider function $g: \mathbb{R}^1 \to \mathbb{R}^3$
$t \rightsquigarrow P + t\hat{u}$

to find derivative of composition $f \circ g: \mathbb{R}^1 \to \mathbb{R}^3 \to \mathbb{R}^1$

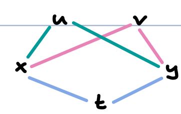
we took $\nabla f \cdot \hat{u}$

derivative of $g = \begin{pmatrix} P_1 + tu_1 \\ P_2 + tu_2 \\ P_3 + tu_3 \end{pmatrix}$, $Dg = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ so we're saying $D(f \circ g) = Df \cdot Dg$ if $h: \mathbb{R}^1 \to \mathbb{R}^2$, then $h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$, so $Dh = \begin{pmatrix} \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial t} \end{pmatrix}$

more generally: multivar. chain rule

given $g: \mathbb{R}^n \to \mathbb{R}^m$, $f: \mathbb{R}^m \to \mathbb{R}^p$ we have $D(f \circ g) = Df \cdot Dg$ (matrix multiplication)

where if $f = \begin{pmatrix} f_1(x_1, \dots, x_m) \\ \vdots \\ f_p(x_1, \dots, x_m) \end{pmatrix}$ then $Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_p}{\partial x_m} \\ \vdots & & & \\ \frac{\partial f_k}{\partial x_k} & \cdots & & \frac{\partial f_p}{\partial x_m} \end{pmatrix}$



EX: let $f: \mathbb{R}^1 \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}^2$
$t \rightsquigarrow \begin{pmatrix} t \\ 3t+2 \end{pmatrix}$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} x+y+3 \\ 2x+5 \end{pmatrix}$

what is $g \circ f(t) = \begin{pmatrix} t + 3t + 2 + 3 \\ 2t + 5 \end{pmatrix}$ b/c $\underline{g \circ f}(t) = g\begin{pmatrix} \overset{x(t)}{t} \\ \underset{y(t)}{3t+2} \end{pmatrix} = \begin{pmatrix} x(t) + y(t) + 3 \\ 2x(t) + 5 \end{pmatrix} = \begin{pmatrix} t + 3t + 2 + 3 \\ 2t + 5 \end{pmatrix} = \begin{pmatrix} 4t + 5 \\ 2t + 5 \end{pmatrix}$

apply g then f

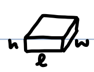
$g(f(t))$

this is LHS of chain rule; we want to compare it to RHS → $D(g \circ f) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

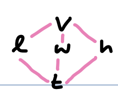
$Df = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\quad$ 2→1 var, so 2x1

$Dg = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ $\quad$ 2→2 var, so 2x2

so $Dg \cdot Df = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

EX: box  ℓ growing @ 0.02 m/min, w growing @ 0.02 m/min, h is shrinking @ -0.02 m/min.

when $\ell = 3$, $w = 2$, $h = 1$, is vol incr or decr, and how fast?



$V = \ell w h$ where ℓ, w, h are functions of t.

$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \ell} \cdot \frac{\partial \ell}{\partial t} + \frac{\partial V}{\partial w} \cdot \frac{\partial w}{\partial t} + \frac{\partial V}{\partial h} \cdot \frac{\partial h}{\partial t}$

another way of expressing this is: $V(\ell,w,h) = \ell wh$. we also have func. $f(t) = \begin{pmatrix} \ell(t) \\ w(t) \\ h(t) \end{pmatrix}$. so composition $V \circ f$ gives volume as function of time.

the formula above is same as $DV \cdot Df$

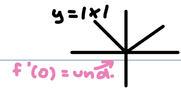
$\ell(t) = 3 + .01t$	$\frac{\partial \ell}{\partial t} = .01$	$\frac{\partial V}{\partial \ell} = wh$
$w(t) = 2 + .02t$	$\frac{\partial w}{\partial t} = .02$	$\frac{\partial V}{\partial w} = \ell h$
$h(t) = 1 - .02t$	$\frac{\partial h}{\partial t} = -.02$	$\frac{\partial V}{\partial h} = w\ell$

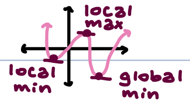
so we get $\frac{\partial V}{\partial t}(0) = 2 \cdot .01 + 3 \cdot .02 + 6 \cdot (-.02) = -.04$ (shrinking)

[3/13/25] - lec. 15
- optimization
- critical pts
- max/min/saddles
- 2nd Deriv. test

Recall: Optimization for $y = f(x)$

UNCONSTRAINED OPTIMIZATION:
- set $f'(x) = 0$, those x's called critical points
- look for pts where $f'(x)$ is undefined
- look @ end behavior as $x \to \pm\infty$
- for critical pts, 2nd. deriv. test to classify
    - $f'' > 0 \to$ local min
    - $f'' < 0 \to$ local max
    - $f'' = 0 \to$ no info (ex: $f(x) = \pm x^n$, $n > 2$)





no global max!

$ax^2 + bx + c = 0$

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + \frac{c}{a} = \frac{b^2}{4a^2}$

$(x + \frac{b}{2a})^2 = \frac{4ac}{4a^2} - \frac{t^2}{4a^2}$

$(x + \frac{b}{2a})^2 = \frac{1}{4a^2}(4ac - b^2)$ ← roots iff $(2a)(2c) - b^2 \geq 0$

CONSTRAINED OPTIMIZATION:

① restrict f to $[a,b]$ $a \leq x \leq b$
- analyze critical pts within interval
- check boundary pts for max/min
- finite # of pts to plug in & find max/min value

② Ex: Find pt on line $x + 3y = 10$ which is closest to origin $(0,0)$.
idea: minimize $\sqrt{\cancel{x^2+y^2}}$ w/ respect to constraint $x + 3y = 10$
(x^2+y^2)
eliminate x, say: $x = 10 - 3y$
minimize $f(y) = (10-3y)^2 + y^2$

back to $z = f(x,y)$:

find local/global max/mins:

now: tan. plane should be horizontal ⇒ look for P where $(\vec{\nabla} f)(P) = \vec{0}$

i.e. find (x,y) pairs so both $f_x = 0$, $f_y = 0$

local mins/max occur @ critical pts. / pts where $\vec{\nabla} f$ = undef

Ex: $f(x,y) = x^2y + xy^2 - 6xy$. find critical pts.

set $f_x = 0$, $\longrightarrow 2xy + y^2 - 6y = 0 \to y(2x + y - 6) = 0$

$f_y = 0 \longrightarrow x^2 + 2xy - 6x = 0 \to x(x + 2y - 6) = 0$

$$\Rightarrow \left\{ \begin{array}{l} y = 0 \text{ OR } 2x + y - 6 = 0 \\ \text{AND} \\ x = 0 \text{ OR } x + 2y - 6 = 0 \end{array} \right] \text{4 possibilities}$$

① $y = 0$ & $x = 0$: crit. pt $(0,0)$
② $2x + y = 6$ & $x = 0$: $y = 6 \longrightarrow (0,6)$
③ $y = 0$ & $x + 2y - 6 = 0$: $x = 6 \to (6,0)$
④ $2x + y = 6$ & $x + 2y = 6 \longrightarrow (2,2)$

are these local max/mins/neither?

NEITHER max or min:

ex: $f(x,y) = x^2 - y^2$

crit. pts @ $(0,0)$ SADDLE

SECOND DERIVATIVE TEST to classify crit. pts:
- for a critical pt. P, compute
$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$
- then: CASES:
    ① $D > 0$, $\left\{ \begin{array}{l} f_{xx} > 0: \text{LOCAL MIN} \\ f_{xx} < 0: \text{LOCAL MAX} \end{array} \right.$
    ② $D < 0$: SADDLE
    ③ $D = 0$: NO INFO

OR $D = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$

$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ ← "Hessian"

IN OUR EXAMPLE:
$f_{xx} = 2y$
$f_{xy} = 2x + 2y - 6 = f_{yx}$
$f_{yy} = 2x$

$D = 4xy - (2x + 2y - 6)^2$

$D(0,0) = -36 \to$ saddle

$D(0,6) = -36 \to$ saddle

$D(6,0) = -36 \to$ saddle

$D(2,2) = 16 - (4 + 4 - 6)^2 = 12$, $f_{xx} = + \to$ local min

FAMOUS Ex: $f(x,y) = x^2y + xy^2$ "monkey saddle"

crit. pt $(0,0)$

$D(0,0) = 0$ NO INFO

look near $(0,0) \to (.1, 0), (.1, -.1)$, etc.

[3/14/25]-lec
- why 2nd D. test is true
- least squares
- constrained optimization

SECOND DERIVATIVE TEST: $z=f(x,y)$ critical pt P, $(\vec{\nabla} f)(P)=\vec{0}$

compute $D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = f_{xx}\cdot f_{yy} - f_{xy}^2$ @ P

IF:
- $D>0$ & $f_{xx}>0 \to P=$ local min
- $D>0$ & $f_{xx}<0 \to P=$ local max
- $D<0 \to P=$ saddle
- $D=0 \to$ no info

WHY does this work?

first, consider $q(x,y)=ax^2+bxy+cy^2$, only crit. pt. $P=(0,0)$

if $4ac-b^2>0 \to$ local min

if $4ac-b^2<0 \to$ saddle

$ax^2+byx+cy^2 = a\left(x^2+\left(\frac{b}{a}y\right)x+\left(\frac{c}{a}y^2\right)\right)$

take half of this, square to "complete square"

$$=a\left[\left(x^2+\frac{b}{a}xy+\left(\frac{b}{2a}\right)^2\right)+\left(\frac{c}{a}y^2\right)-\left(\frac{b}{2a}y\right)^2\right] = a\left[\left(x+\frac{b}{2ay}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)y^2\right] = a\left[\left(x+\frac{b}{2a}y\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)y^2\right]$$

notice $q_x = 2ax+by$, $q_y = bx+2xy$

$q_{xx}=2a$

$q_{xy}=b$

$q_{yy}=2c$

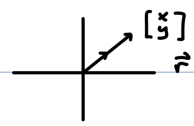
$D=q_{xx}q_{yy}-q_{xy}^2$

to generalize, compute Taylor poly. @ critical pt. get some formula

How are saddles "oriented"?

back to $q(x,y)=ax^2+bxy+cy^2$ ← Hessian $\begin{bmatrix} q_{xx} & q_{xy} \\ q_{yx} & q_{yy}\end{bmatrix}$

notice $(\vec{\nabla} q)\left(\begin{bmatrix} x \\ y\end{bmatrix}\right) = \begin{bmatrix} 2a & b \\ b & 2c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = H\begin{bmatrix} x \\ y\end{bmatrix}$



$(\vec{\nabla} q)(\vec{x}) = H\vec{x}$ say $\vec{v}$ is eigenvect: $(\vec{\nabla} q)(\vec{v}) = H\vec{v} = \lambda\vec{v}$

$\lambda>0$: grad points away

$\lambda<0$: grad points towards

the eigenvect. direction tells you how the saddle is oriented in space

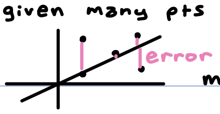
EX: $f(x,y)=x^2+xy^2-bxy$ @ crit pt P

$H(0,6) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 6 & 0\end{bmatrix}$

find eigenvals/vects @ $(0,0) \to H=\begin{bmatrix} 0 & -6 \\ -6 & 0\end{bmatrix}$

LEAST SQUARES / BEST FIT LINE:

given many pts $(x_1,y_1)\ldots(x_n,y_n)$



minimize $E=\sqrt{\text{error}_1^2+\ldots+\text{error}_n^2}$

GOAL: find m,b s.t. $y=mx+b$ is "best fit"

minimize $f(m,b)=(y_1-(mx_1+b))^2+(y_2-(mx_2+b))^2)-\ldots+(y_n(mx_n+b)^2)$

2 var optimization problem

$\begin{bmatrix} mx_1+b \\ \vdots \\ mx_n+b\end{bmatrix} = m\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix} + b\underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1\end{bmatrix}}^{\vec{1}}$

minimizing $|\vec{y}-(m\vec{x}+\vec{1})|$

distance $=\sqrt{\Sigma \text{ squares}}$

[3/17/25]-rec

1) find distance from origin to plane $x+2y+2z=12$

→ trying to minimize $\sqrt{a^2+b^2+c^2}$ for (a,b,c) satisfying $a+2b+2c=18$

notice $18-2b-2c$, so we're trying to minimize $f(b,c)=\sqrt{(18-2b-2c)^2+b^2+c^2}$

it's equally good to minimize $g(b,c)=(18-2b-2c)^2+b^2+c^2$

find critical pts: where $\vec{\nabla} g = \left(\frac{\partial g}{\partial b}, \frac{\partial g}{\partial c}\right)=0$

$\frac{\partial g}{\partial b} = 2\cdot -2(18-2b-2c)+2b = 10b+8c-72=0$

$\frac{\partial g}{\partial c} = -2\cdot 2(18-2b-2c)+2c = 8b+10c-72=0$

$2b-2c=0 \to b=c$

$b=c=4$

one C.P. @ $(4,4)$.

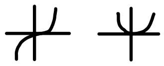
to figure out which type, find det. of Hessian matrix $\det\begin{pmatrix}10 & 8\\ 8 & 10\end{pmatrix} = 36$

then det > 0 → either local max/min. $g_{bb} > 0$ → <u>local min</u> @ $(b,c) = (4,4)$ so $(a,b,c) = (2,4,4)$ @ min dist = $\sqrt{2^2+4^2+4^2} = 6$

once we have CP:

	det>0	det<0	det=0
$f_{xx} > 0$	min	saddle point	test inconclusive
$f_{xx} < 0$	max		
$f_{xx} = 0$	?		

↪ think one var. $y = x^3$, $y = x^4$, both have $y''(0) = 0$



if i wanted to find max/min of some $f(x,y)$:

- if min/max in the interior of region, it's a critical pt that is a local min/max.
- it also could be on boundary: look @ function on each edge:
  - bottom edge: $f(x,0)$ & find extrema for $x \in [0,1]$
  - other edges: similar
  - check corners

combining pts from interior, boundary, and corners, get a list of possible extremes & evaluate function @ those values to find critical max/mins over the region

find the minimum of $g(x,y) = x^3 - y^3 - 9xy + 8$

observation: when $x \to \infty$ & $y \to \infty$, then $g(x,y) \to \infty$ (no max)

when $x \to \infty$ & $y \to \infty$, then $g(x,y) \to -\infty$ (no min)

2b) find min/max vals of $g(x,y)$ on the square $[0,1] \times [0,1]$

find C.P.: $\left.\begin{aligned}\frac{\partial g}{\partial x} &= 3x^2 - 9y = 0\\ \frac{\partial g}{\partial x} &= -3y^2 - 9x = 0\end{aligned}\right\} \rightarrow \begin{aligned}x^2 &= 3y\\ -y^2 - 3x &= 0\end{aligned} \rightarrow -\left(\frac{x^2}{3}\right)^2 - 3x = 0$

$-\frac{x^4}{9} - 3x = 0$ → either $x = 0$ or $x = 3$

our critical points are $(0,0)$ or $(-3, 3)$

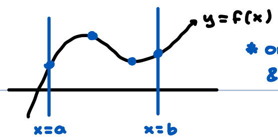
so we don't have any critical points in the interior of the square

next, check the edges:

- bottom edge $(y = 0,\ 0 \le x \le 1)$: $x^3 + 8$ – no critical pt in middle of the edge
- left edge: $(x = 0,\ 0 \le y \le 1)$: $-y^3 + 8$ – no C.P. in the middle
- top edge: $(y = 1,\ 0 \le x \le 1)$: $x^3 - 1 - 9x + 8$ — C.P. are 0's of $3x^2 - 9 = 0 \to x = \pm\sqrt{3}$ → not on edge
- right edge: $(x = 1,\ 0 \le y \le 1)$: $1 - y^3 - 9y + 8$ — C.P. $-3y^2 - 9 = 0 \to y = \pm\sqrt{3i}$ → no C.P. (no real #s)

so, the max/min occur on one of 4 corners

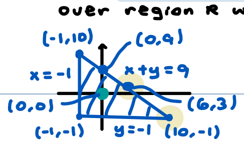
[3/18/25]-lec
- constrained optimization
- regions/bounds
- lagrange multipliers



* only have to check critical points & endpoints

Ex: find max/min values of $f(x,y) = x^2y$ over region R

over region R which is vertices of triangle w/ vertices $(-1,10), (-1,-1), (10,-1)$



<u>IDEA:</u> find critical pts inside the region &
- analyze the boundary
- plug all interesting pts into f
- only critical pt: $(0,0)$

BOUNDARIES made up of 3 line segments

$x = -1$: $g(y) = f(-1,y) = (-1)^2 y = y$, $-1 \le y \le 10$

$g'(y) = 1$, no critical pts, so just look @ boundary pts $(-1,-1), (-1,10)$

$y = -1$: $h(x) = f(x,-1) = -x^2$, $-1 \le x \le 10$

$h'(x) = -2x = 0$ when $x = 0 \to (0,-1)$

$x + y = 9$: $f(x,y) = x^2 y = x^2(9-x) = 9x^2 - x^3 = p(x)$, $-1 \le x \le 10$

$p'(x) = 18x - 3x^2 = 3x(6-x) = 0$

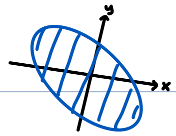
$y = 9 - x$: $(0,9), (6,3)$ and endpoints $(-1,10), (10,-1)$

<u>PLUG IN:</u>
- $f(0,0) = 0$
- $f(-1,-1) = -1$
- $f(10,-1) = -100$
- $f(-1,10) = 10$
- $f(0,-1)$ 0
- $f(0,9) = 0$
- $f(6,3) = 108$

min is -100 @ $(10,-1)$

max is $+108$ @ $(6,3)$

Ex: find max/min values of $f(x,y) = xy$ on region R defined by $x^2+xy+y^2 \le 3$



INSIDE: C.P. of $f \to (0,0)$ — * can't isolate variable easily

BOUNDARY: $g(x,y) = x^2+xy+y^2 = 3$ — level curve of g

LAGRANGE MULTIPLES: max/min $f(x,y)$ with respect to constraint $g(x,y) = c$ (constraint func; level curve)

it turns out: @ max/min pts of f, $\vec\nabla f$ is parallel to $\vec\nabla g$.

$(\vec\nabla f = \lambda \cdot \vec\nabla g)$ — lambda

IN the example: set gradients parallel & try to find pts where this happens

$\vec\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}$ $\quad \vec\nabla g = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix}$

get $y = \lambda(2x+y)$
$x = \lambda(x+2y)$] lagrange — find x, y, λ
$x^2+xy+y^2 = 3$] constraint

solve: $\lambda = \frac{y}{2x+y} = \frac{x}{x+2y}$ → have to think ab. $x=0, y=0$ $\Rightarrow f = 0$

$\Rightarrow x(2x+y) = y(x+2y)$
$\Rightarrow 2x^2+xy = xy+2y^2$
$\Rightarrow x^2 = y^2 \to y = \pm x$

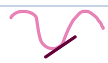
TWO CASES:

$y = x$: $x^2+xy+y^2 = 3$
$x^2 = 1 \to x = \pm 1$
$(1,1), (-1,-1)$
$xy = 1$ max

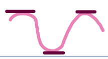
$y = -x$: $x^2 + x(-x) + (-x)^2 = 3$
$x = \pm\sqrt{3}$
$(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$
$xy = -3$ min

[3/19/25] - rec

LAGRANGE MULTIPLIERS:

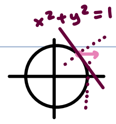


- what is a total derivative
- what is 1st derivative test - finding extrema ⇝ $\vec\nabla f = 0$
- why lagrange multipliers is a geometric slope
  - f maximize ⇝ "profit"
  - g constraint ⇝ "cost"



Ex: $g(x,y) = 0$
$\{ x^2+y^2 = 1$

$x^2+y^2=1$



$\langle 2x, 2y \rangle$
$\{ \langle 2,0 \rangle$

$\vec\nabla f = \lambda \vec\nabla g$
$F = x$
$\langle 1,0 \rangle$

Ex: $g(x,y,z) = 0$

$z = \_\_$

$x + 2y + 2z = 18$

what is distance to the origin?

minimize: $x^2+y^2+z^2$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1,2,2 \rangle$

$2x = \lambda$
$2y = 2\lambda$
$2z = 2\lambda$
$\boxed{2x = y = z}$

$x + 2(2x) + 2(2x) = 18$
$x = 2, y = 4, z = 4$
pt that satisfies eq.
dist $= \sqrt{2^2+4^2+4^2} = \boxed{6}$

Ex: find external values of $F = x^2y^2z^2$ subject to $g = x^2+4y^2+9z^2-27$ $\{g=0\}$

$\vec\nabla F = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle = \lambda \langle 2x, 8y, 18z \rangle$ (4, 9)

$xy^2z^2 = x\lambda \to$ $x=0$ OR $\lambda = y^2z^2$ — account!
$yx^2z^2 = 4y\lambda \to$ $y=0$ OR $\lambda = \frac{x^2z^2}{4}$
$zx^2y^2 = 9z\lambda \to$ $z=0$ OR $\lambda = \frac{x^2y^2}{9}$

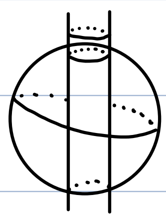
$y^2 = \frac{x^2}{4}$

$\lambda = \frac{x^4}{36}$

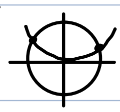
$9z^2 = 27 - 2\lambda^2 \to \lambda = \frac{x^2}{4}\left(3 - \frac{2x^2}{9}\right)$

$\frac{x^4}{36} = \frac{x^2}{4}\left(3 - \frac{2x^2}{9}\right)$ — * lots of +/− in the solutions (6 sols.)

Ex:



$\nabla F = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$
$\begin{cases} g_1 = 0 \\ g_2 = 0 \end{cases}$



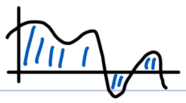
Ex: highest & lowest pts (F = z) $x+y+z = 12$ & $z = x^2+y^2$

$\vec\nabla F = \langle 0,0,1 \rangle = \lambda_1 \langle 1,1,1 \rangle + \lambda_2 \langle 2x, 2y, -1 \rangle$

[3/20/25] - lec

INTEGRATION:

recall: single integral of $y=f(x)$, $a \le x \le b$

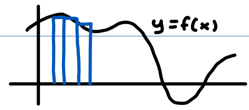


$\int_a^b f(x)dx =$ "signed" area b/t graph & x axes

How to compute?

IDEA: estimate w/ Riemann Sum i.e. rectangles

then shrink the width of rect → 0, in the limit you get exact area.



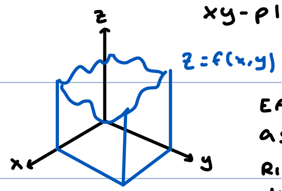
RIEMANN SUM = Σ(areas of rectangles)

$$= \sum_{i=n}^{n} \underbrace{f(x_i)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{base}} \quad \xrightarrow{\Delta x \to 0} \quad \int_{x=a}^{x=b} f(x)dx$$

from thm of calc = (take anti-deriv. of f., plug in endpoints, and subtract)

$$(f = F') = F(b) - F(a) = F\Big|_a^b$$

DOUBLE INTEGRAL: given $z=f(x,y)$ over region R in (x,y) plane, find volume "under" surface & "above" xy-plane.

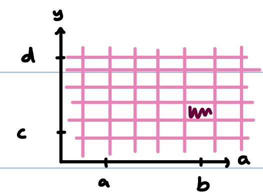


EASIEST CASE: R is a rectangle, $a \le x \le b$, $c \le y \le d$

RIEMANN SUM:

divide $[a,b]$ into Δx pieces

$[c,d]$ into Δy pieces



call this $\Delta A = \Delta x\, \Delta y$

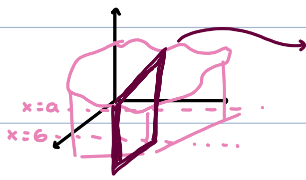
pick (x_A, y_A) inside each subrectangle

base: ΔA

height: $f(x,y)$

RIEMANN: $\sum_{i=1}^{n}$ (vol. of each rect. box)

$$= \sum_{i=1}^{n} \underbrace{f(x_i, y_i)}_{\text{height}} \underbrace{\Delta A}_{\text{base}} = \int_R f(x,y)\,dA = \iint_R f(x,y)\,dA$$



thicken up by Δy

area of slice $= \int_a^b f(x,y)dx$ ← for each y, $c \le y \le d$, we get a slice of area $A(y)$ ← depends

$y=y$, slice of surface $z=f(x,y)$, $a \le x \le b$

VOLUME OF THICKENED-UP SLICE $= \underbrace{A(y)}_{\substack{\text{area of func. of} \\ \text{slice}}} \underbrace{dy}_{\text{thickness}}$

TOTAL VOL: add slices from $y=c$ to $y=d$

$$= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy$$

EX: $z = xy + x$, R: $1 \le x \le 3$

$2 \le y \le 5$

vol. under $f(x,y)$ above R $= \int_{y=2}^{y=5} \left(\int_{x=1}^{x=3} (xy+x)dx \right) dy$

AREA $\quad A(y) = \int_{x=1}^{x=3} (xy+x)dx = \left[\frac{1}{2}x^2 y + \frac{1}{2}x^2\right]_1^3 = 4y+4$

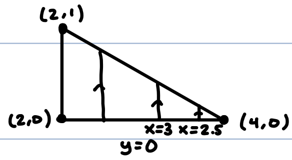
Vol $\quad = \int_{y=2}^{y=5} (4y+4)dy = \left[2y^2 + 4y\right]_2^5 = (50+20) - (8+8)$

SWITCH ORDER: $\int_{x=1}^{x=3} \left(\int_{y=2}^{y=5} (xy+x)dy \right) dx$ ← works either way!

(3/21/25) - lec

integrate over more complicated regions than rectangles

Ex: $f(x,y) = x+y$ over triangular region



we want $\int_R f(x,y)\,dA = \underbrace{\int_{x=}^{x=}}_{\text{area of all slices}} \Big(\underbrace{\int_{y=}^{y=} (x+y)\,dy}_{\text{area of slice}} \Big)\,dx$

fix values of x, see what slice looks like

$x=3$: $\int_{y=0}^{y=\frac{1}{2}} (x+y)\,dy$ — the slices are variable.

$x=2$: ... etc.

x = constant slice

$\int_{y=0}^{y=2+\frac{1}{2}x} (x+y)\,dy$

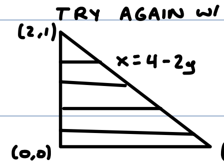
$\int_R (x+y)\,dA = \int_{x=2}^{x=4} \Big(\int_{y=0}^{y=2-\frac{1}{2}x} (x+y)\,dy \Big)\,dx = \int_{x=2}^{x=4} \Big(-\frac{3}{8}x^2 + x + 2 \Big)\,dx = -\frac{3}{8}\cdot\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big|_2^4 = \boxed{3}$

leftmost slice

$xy + \frac{1}{2}y^2 \Big|_0^{2-\frac{1}{2}x}$

$= -\frac{3}{8}x^2 + x + 2$

TRY AGAIN w/ y = const. slices



left endpt. of each slice: x=2

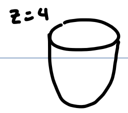
right endpt of each slice: x=4-2y

b/t slice: y=0 → y=1

will have the SAME RESULT!

$\int_R (x+y)\,dA = \int_{y=0}^{y=1} \Big(\int_{x=2}^{x=4-2y} (x+y)\,dx \Big)\,dy$

Ex:

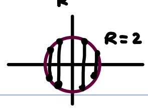


$z = x^2+y^2$

find volume above $z = x^2+y^2$ but below $z=4$

$V = \int_R \big(4-(x^2+y^2)\big)\,dA = ?$

intersect $z=4$ with $z=x^2+y^2$ & we get $x^2+y^2=4$, circle of rad 2



$x^2+y^2=4 \rightarrow y = 4-x^2$

$y = \pm\sqrt{4-x^2}$

bottom: $y = -\sqrt{4-x^2}$

top: $y = +\sqrt{4-x^2}$

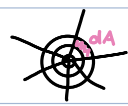
right: $x=2$

left: $x=-2$

$\int_{x=2}^{x=4} \Big(\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \big(4-(x^2+y^2)\big)\,dy \Big)\,dx$ ← SWITCH TO POLAR!

POLAR: $x = r\cos\theta$ $\quad r^2 = x^2+y^2$

$y = r\sin\theta$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$



$dA = r\,dr\,d\theta$

R in Polar: $0 \le r \le 2$ $\quad$ but what is dA?

$0 \le \theta \le 2\pi$

$= \int_{\theta=0}^{\theta=2\pi} \Big(\int_{r=0}^{r=2} (4-r^2)\,r\,dr \Big)\,d\theta = \int_{\theta=0}^{\theta=2\pi} 4\,d\theta = 8\pi$

[4/1/25] - lec

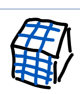
- triple ints
- area & vol. reinsited
- vols. of complicated regions

TRIPLE INTS:

- Similar to double ints but integrating over $\underbrace{\text{3D space}}_{T}$ instead of 2D region R

Ex: integrate $f(x,y,z) = x^2y + yz$ over rectangular box $\underbrace{1 \le x \le 3,\ -2 \le y \le 4,\ 2 \le z \le 5}_{T}$

$\int_T f(x,y,z)\,dV$ ← comes from Riemann Sum $\sum_{i=1}^{n} f(P_i)\Delta V_i$



compute as iterated integral

$= \int_{x=1}^{x=3} \int_{y=-2}^{4} \int_{z=2}^{5} (x^2y + yz)\,dz\,dy\,dx \quad = \Big[x^2yz + \frac{1}{2}yz^2 \Big]_2^5 = \Big(5x^2y + \frac{25}{2}y\Big) - \Big(2x^2y + 2y\Big) = 3x^2y + \frac{21}{2}y$

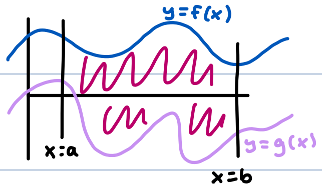
$= \int_{x=1}^{x=3} \int_{y=-2}^{4} \Big(3x^2y + \frac{21}{2}y\Big)\,dy\,dx = \Big[\frac{3}{2}x^2y^2 + \frac{21}{4}y^2\Big]_{-2}^{4} = \ldots\ 18x^2 + 63$

$\int_{x=1}^{3} 18x^2 + 63\,dx = \Big[6x^3 + 63x\Big]_1^3 = \ldots = \boxed{282}$

*variables should dissapear!

CHECK:
dydzdx
dzdxdy
⋮
diff. orders

Areas Revisited:

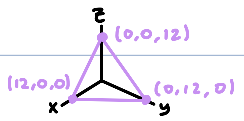


$$\text{Area} = \int_{x=a}^{b} (f(x)-g(x))\,dx = \int_{x=a}^{b}\int_{y=g(x)}^{y=f(x)} 1\,dy\,dx$$

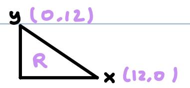
$$\int_R 1\,dA = \int_R dA$$

Volumes Revisited:

Ex: volume under plane $x+y+z=12$, $x,y,z \geq 0$



space region T



as double int: $V = \int_R (12-x-y)\,dA$

as triple int: $V = \int_R \int_{z=0}^{z=12-x-y} 1\,\underbrace{dz\,dA}_{dV} = \int_T 1\,dV$

A More Complicated Space Region:

Ex: find the Vol. of space region T bounded by

$\underbrace{z=x^2,\ y+z=10}_{\text{intersect these}}$, $y=0$ vertical wall

1) find shadow of T in xy-plane

→ shadow made up of: (a) vertical walls ($\perp$ to xy-plane, no "z"s)

(b) intersections of "z =" equations

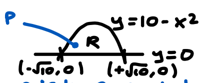
$y=0$, $z=x^2$, $z=10-y$ → $x^2=10-y$ → $y=10-x^2$

then $V = \int_R (z_{top} - z_{bot})\,dA = \int_R \int_{z_{bot}}^{z_{top}} dz\,dA$

2) write this down

3) identify top & bottom

we know shadow: $y=0$ & $y=10-x^2$



pick a point inside R & find which eqn produces greater value (top)

$P = (0,1)$

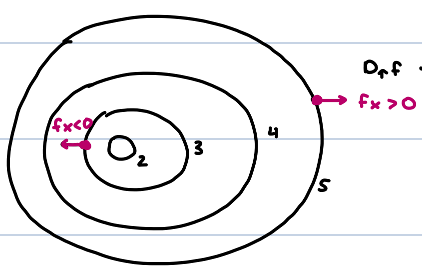
$y=10$, $y=0$

So $\text{Vol} = \int dV = \int_R \left(\int_{z=x^2}^{z=10-y} dz \right) dA = \int_{x=-\sqrt{10}}^{x=\sqrt{10}} \int_{y=0}^{y=10-x^2} \int_{z=x^2}^{z=10-y} dz\,dy\,dx$

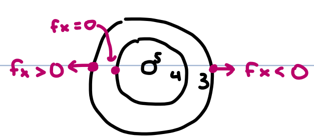
[4/2/25] - office hrs

$x^n = ?$ always has n solutions b/c if $r_i e^{i\theta_i}$ is a solution $r_i e^{i\theta_i} \cdot e^{i2\pi k/n}$ is always a solution

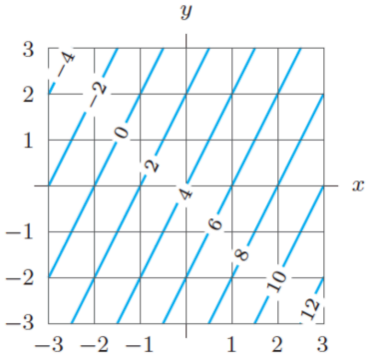
Ex:



$D_{\vec{u}}f$ "derivative of f in → direction"



EX:



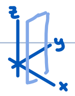
$f(x,y) = ax+by+c$

@ $(0,0) \to f(0,0) = 4 \to c = 4$

@ $(-1,2) \to f(-1,2) = 0 \to -a+2b+4=0$

Ex: a surface whose x = constant cross-sections are lines but which is not a plane.

$g(17,y,z) = ay+bz+c$



$= a(17)y + b(17)y + c(17)$

$g(x,y,z) = a(x)y + b(x)z + c(x)$

if $a(x)$, $b(x)$, $c(x)$ are constant functions, then we would get a plane

POSSIBLE NON-PLANE CHOICE: $a(x) = x^2+1$, $b(x) = \cos(5x)$, $c(x) = 18$

if $g(x,y,z) = (x^2+1)y + \cos(5x)z + 18$

if $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ exactly when $r_1 = r_2$

$e^{i\theta} = \cos\theta + i\sin\theta$ for <u>any</u> θ, not just $\theta \in [0, 2\pi)$

$(re^{i\theta})^n = r^n (e^{i\theta})^n = r^n e^{in\theta}$

"

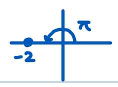
$(r\cos\theta + i r\sin\theta)^n$ $\quad e^{i\theta} \cdot e^{i\theta} = e^{i2\theta}$

EX: find the 4th roots of -2

$-2 + 0i$

$r = \sqrt{-2^2} = 2$

$\theta = \tan^{-1}\left(\frac{0}{-2}\right) = \pi$



*CAREFUL! Sign

$2e^{i\pi} = 2\cos\theta + i2\sin\theta$

$\sqrt[4]{2}e^{i(\pi+2\pi k)/4}$

$k=0$: $2e^{i\pi/4} =$

$\sqrt[4]{2}\, e^{i\pi/4}$, $\sqrt[4]{2}\, e^{i3\pi/4}$, $\sqrt[4]{2}\, e^{i5\pi/4}$, $\sqrt[4]{2}\, e^{i7\pi/4}$

solve $(re^{i\theta})^4 = 2e^{i\pi}$
$= r^4 e^{i4\theta}$

then $\pi = 4\theta + 2\pi k$ for some integer k

$\theta = \frac{\pi - 2\pi k}{4} = \frac{\pi}{4}, \frac{\pi}{4} \pm \frac{\pi}{2}, \frac{\pi}{4} \pm \pi, \ldots$

4 roots: $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}$
DISTINCT

ALSO, $r^4 = 2 \to r = \sqrt[4]{2}$ (the positive real one)

<u>GLOBAL / LOCAL min/max:</u>

- for each CP → find local min/max & saddles
  - compare local mins & local maxes
  - see which is global
- if no CP: $\vec{\nabla} f \neq 0$
- specifically for saddle pts, there is upward & downward parabola

one λ points to each

$\lambda > 0$, ∇f points away from CP

$\lambda < 0$, ∇f points towards CP

[4/2/25] - recitation (TEST PREP)

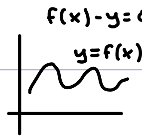
TOTAL DERIVATIVE: a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is given by $\begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$

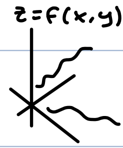
then the total derivative $Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$ has all info about partial derivatives of f.

SPECIAL CASE: $f: \mathbb{R}^n \to \mathbb{R}^1$.

then **DIRECTIONAL DERIVATIVE** $D_{\hat{u}} f$ = "derivative of f along a path in direction $\vec{u}$"

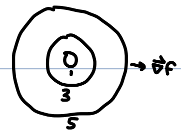
$= \vec{\nabla} f \cdot \hat{u} = |\vec{\nabla} f| \cos\theta$





so $\vec{\nabla} f$ is in direction of "steepest increase"

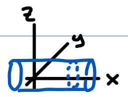
& level curves are $\perp$ to $\vec{\nabla} f$.



$f(x,y)$ = height @ (x,y)

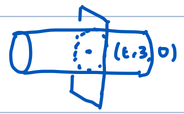
SURFACES:

EX: cylinder, axis is line through $(0,3,0)$ para. to x-axis, $r=5$



$(t, 3, 0)$ ← y & z stay fixed, but x changes (along axis)

take the cross-section $x=t$ & intersection of cylinder w/ that cross sec. is a circle of $r=5$ around $(t,3,0)$ in cross section



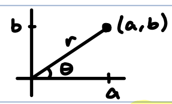
$(y-3)^2 + z^2 = 25$ when $x=t$

so we get $(y-3)^2 + z^2 = 25$

COMPLEX #'S:

RECTANGULAR COORDS: $z = x+iy$

POLAR COORDS: $z = re^{i\theta}$



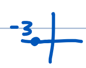
convert: given rectangular, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

given polar: $x = r\cos\theta$, $y = r\sin\theta$

FINDING nth roots of cx #:

EX: find cube roots of -3

CONVERT -3 to polar:



$r = \sqrt{-3^2} = 3$

$\theta = \tan^{-1}\left(\frac{0}{-3}\right) = \pi$

REVIEW!

$3e^{i\pi}$

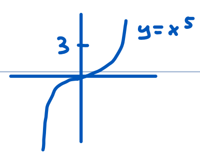
want to find r, θ s.t. $(re^{i\theta})^3 = 3e^{i\pi}$

$= r^3 e^{3i\theta}$ → then $r^3 = 3$ & $3\theta = \pi + k2\pi$ for some int k

$r = \sqrt[3]{3}$ $\theta = \frac{\pi + 2\pi k}{3}$

$\sqrt[3]{3}\, e^{i\left(\frac{\pi + 2\pi k}{3}\right)}$ where k is some int.

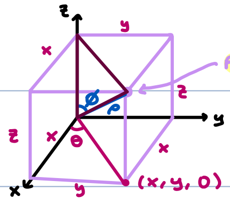
$\sqrt[3]{3}\, e^{i\pi/3}$, $\sqrt[3]{3}\, e^{\pi}$, $\sqrt[3]{3}\, e^{5\pi/3}$, $\sqrt[3]{3}\, e^{7\pi/3}$



[4/8/25] - lecture 21

- cylindrical & spherical coords
- triple ints in cylindrical & spherical

OTHER COORDINATE SYSTEMS:



dist. from z-axis

$P(x,y,z) = (r, \theta, z) = (\rho, \phi, \theta)$

cylindrical = "polar + z"

spherical (latitude, longitude)

ρ = rho = dist. from origin

ϕ = phi = "fee" = angle the point makes with positive z-axis

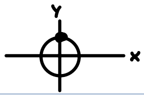
(ϕ = "fie")

CYLINDRICAL: $0 \le r < \infty$, $0 \le \theta < 2\pi$, $-\infty < z < \infty$

SPHERICAL: $0 \le \rho < \infty$, $0 \le \phi \le \pi$ (latitude goes to 180°), $0 \le \theta \le 2\pi$

$\phi = 0$ "north pole"
$\phi = \pi$ "south pole"

EX: (0,5,0) in rectangular

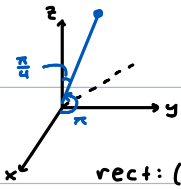


cylindrical: $r = 5$, $\theta = \frac{\pi}{2}$, $z = 0$, $(5, \frac{\pi}{2}, 0)$

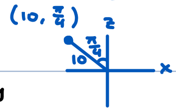
spherical: $(5, \frac{\pi}{2}, \frac{\pi}{2})$

Ex: (0,-5,0) cylindrical: $(5, \frac{3\pi}{2}, 0)$ spherical: $(5, \frac{\pi}{2}, \frac{3\pi}{2})$

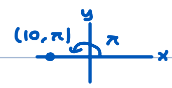
Ex: $(10, \frac{\pi}{4}, \pi)$ in spherical - what are rectangular coordinates?



side view:

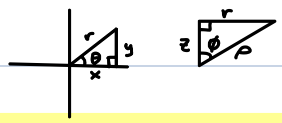


top view:



rect: $(-5\sqrt{2}, 0, 5\sqrt{2})$

CONVERSION FORMULAS:



$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$z = \rho\cos\phi$
$r = \rho\sin\phi$
$\rho = \sqrt{z^2+r^2}$
$\phi = \tan^{-1}\left(\frac{r}{z}\right)$

RECTANGULAR ⟷ CYLINDRICAL

same as polar $(x,y) \leftrightarrow (r,\theta)$
$z = z$

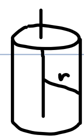
RECTANGULAR ⟷ SPHERICAL

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$\rho = \sqrt{x^2+y^2+z^2}$
$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

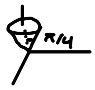
Some surfaces in cylindrical/spherical:

r = constant: cylinder with axis = z-axis



ρ = constant: sphere centered @ (0,0,0)

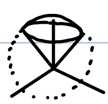
ϕ = constant: cone! ex: $z = \sqrt{x^2+y^2}$, $\phi = \frac{\pi}{4}$



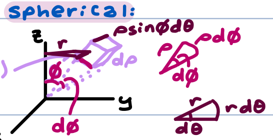
exception: $\phi = \frac{\pi}{2}$ xy-plane!

INTEGRATION:

Ex: ice cream cone
volume of region above cone $z = \sqrt{x^2+y^2}$ but inside sphere $x^2+y^2+z^2 = 9$



dV in other coordinates:
(reminder: in polar, $dA = r\,dr\,d\theta$)
in cylindrical: $dV = r\,dr\,d\theta\,dz$
in spherical:



$= \rho\sin\phi\,d\theta$

ICE CREAM CONE IN SPHERICAL:

$0 \le \rho \le 3$
$0 \le \phi \le \frac{\pi}{4}$
$0 \le \theta \le 2$

i.e. triple int has constant bounds

$$V = \int_T dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{3} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

LITTLE BOX OF DIMENSIONS

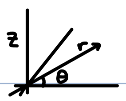
$d\rho \times \rho\,d\phi \times \rho\sin\theta$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4}\sin\phi\,d\phi\right)\left(\int_0^3 \rho^2\,d\rho\right) = 2\pi\cdot\left(-\cos\phi\Big|_0^{\pi/4}\right)\cdot\left(\tfrac{1}{3}\rho^3\Big|_0^3\right) = \boxed{2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot 9}$$

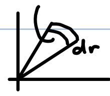
[4/9/25] - recitation

Cylindrical Coords:



(z, r, θ)

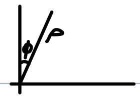
replace x & y by polar coords



area element in polar coords is $r\,dr\,d\theta$

VOLUME ELEMENT: $r\,dr\,d\theta\,dz$

Spherical Coords:



(ρ, ϕ, θ)

ρ = distance from origin

ϕ = angle to north pole (latitude)

$(0 \le \phi \le \pi)$

θ = angle in the xy plane (same as cylindrical)

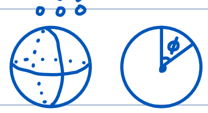
$(0 \le \theta \le 2\pi)$

VOLUME ELEMENT: $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Ex: compute volume of a sphere of radius A in spherical coords

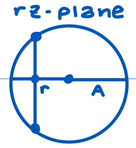
$\rho \le A$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{A} 1\cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{3}A^3 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \tfrac{1}{3}A^3\left[-\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} \tfrac{1}{3}A^3\left[1-(-1)\right] d\theta = \int_0^{2\pi} \tfrac{2}{3}A^3\, d\theta = \boxed{\tfrac{4\pi}{3}A^3}$$



TRY SAME calculation, but cylindrical coords:

rz-plane



$r_0^2 + z^2 = A^2$

$\Rightarrow z = \pm\sqrt{A^2 - r_0^2}$

$$\int_0^{2\pi}\int_0^{A}\int_{-\sqrt{A^2-r^2}}^{\sqrt{A^2-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{A} 2r\underbrace{\sqrt{A^2-r^2}}_{u^{1/2}}\, dr\, d\theta = \int_0^{2\pi}\int_{A^2}^{0} -\sqrt{u}\, du\, d\theta = \int_0^{\pi} \frac{-u^{3/2}}{3/2}\Big|_{A^2}^{0} d\theta$$

* r-coord must be ⊕!

$$= \int_0^{2\pi} \frac{-(A^2-r^2)^{3/2}}{3/2}\Big|_{A^2}^{0} d\theta = \int_0^{2\pi} \frac{A^3}{3/2}\, d\theta = \frac{2\pi A^3}{3/2} = \boxed{\frac{4\pi}{3}A^3}$$

*non-monotone is hard!

EX: suppose hemisphere radius 5 w/ density $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}$.

what is its volume?

want to integrate 1/r over the hemisphere in cylindrical coords

$r^2 + z^2 \le 5^2$ and $z \ge 0$

r^2

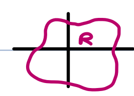
$$\int_0^{5}\int_0^{\sqrt{25-z^2}}\int_0^{2\pi} \frac{1}{r}\, r\, d\theta\, dr\, dz = \int_0^{5}\int_0^{\sqrt{25-z^2}}\int_0^{2\pi} d\theta\, dr\, dz$$

[4/10/25] - LECTURE 22

- applications of integration
- change of vars in multiple integrals

APPLICATIONS OF INTEGRATION:

① **AVERAGE VALUE:** say $z = f(x,y)$, looking over some region R.



what is avg. value of some region f on R?

pick pts. in R. Avg $\approx \dfrac{f(x_1,y_1) + \dots + f(x_n,y_n)}{n}$

as $n \to \infty$? (cover every point in R)

more systematic: break up R into pieces of size ΔA, pick one point (x_i, y_i) in each subregion.

get "Riemann Sum": $\sum_{i=1}^{n} f(x_i,y_i)\cdot \frac{1}{n}$

notice: $n\cdot\Delta A = (\text{Area of } R) \to \frac{1}{n} = \frac{\Delta A}{\text{Area of } R}$

plug in, get $f_{avg} \approx \frac{1}{\text{Area of } R}\sum_{i=1}^{n} f(x_i,y_i)\,\Delta A$

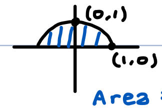
as $\Delta A \to 0$, get $f_{avg} = \frac{1}{\text{Area of } R}\cdot \int_R f(x,y)\,dA$

For $y = f(x)$ on $[a,b]$,

$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

integral of function f over geometric thing $G = \int_G f\,d = \begin{pmatrix}\text{avg. val of}\\ f \text{ over } g\end{pmatrix}\begin{pmatrix}\text{size of}\\ G\end{pmatrix}$

Ex: let U = upper half of disk $x^2+y^2 \le 1$. What is avg. y-val of pt. in U?
(a little less than $\frac{1}{2}$ based on logic)



Area $= \frac{1}{2}\pi$

$y = r\sin\theta$

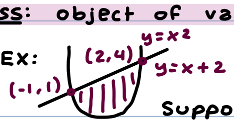
$$y_{avg} = \frac{1}{\text{Area of } U}\int_U y\,dA = \frac{\pi}{2}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{1}(r\sin\theta)\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}\Big(\underbrace{\int_0^\pi \sin\theta\,d\theta}_{2}\Big)\Big(\underbrace{\int_0^1 r^2\,dr}_{\frac{1}{3}}\Big) = \frac{4}{3\pi} \approx \boxed{0.42}$$

Avg. x-value of pts on U? 0 by symmetry!

② **Mass:** object of variable density. integrate to find mass

Ex:



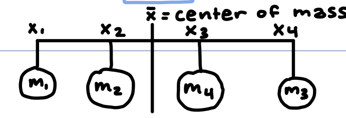
Suppose R has density $\delta(x,y) = kx^2$ $\left[\frac{g}{m^2}\right]$
(per unit area)

$$\text{Mass} = \int_R \delta(x,y)\,dA \qquad (g/m^2)(m^2) = g$$

$$\text{Mass} = \int_{-1}^{2}\int_{x^2}^{x+1} kx^2\,dy\,dx = \ldots = \boxed{\frac{63}{20}k}$$

③ CENTER OF MASS

$\bar{x}$ = center of mass = balance pt.



$\sum m_i(\bar{x}-x_i) = 0$

$\sum m_i\bar{x} = \sum m_i x_i$

$\rightarrow \bar{x} = \frac{\sum m_i x_i}{\sum m_i}$ ← total mass

GENERALIZE: for an object R of density $\delta(x,y)$, the x-coord of center of mass is

$$\bar{x} = \frac{\int_R x\,dm}{\int_R dm} = \frac{\int_R x\cdot\delta(x,y)\,dA}{\int_R \delta(x,y)\,dA}$$

$\bar{x} = \int x\cdot$

Ex: find CoM of object from before

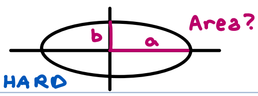
$$\bar{x} = \frac{\int_R x(kx^2)\,dA}{\int_R kx^2\,dA} = \frac{\int_{-1}^{2}\int_{x^2}^{x+2} kx^3\,dy\,dx}{\frac{63}{20}k} = \boxed{\frac{8}{7}}$$

$$\bar{y} = \frac{\int_R y(kx^2)\,dA}{\frac{63}{20}k} = \boxed{\frac{118}{49}}$$

→ so center of mass is $\left(\frac{8}{7}, \frac{118}{49}\right)$

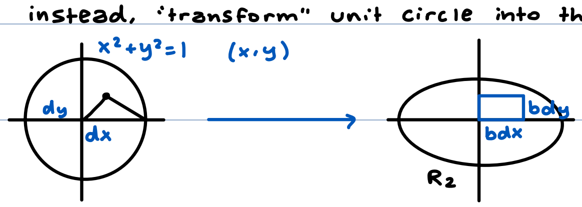
CHANGE OF VARIABLES IN MULTIPLE INTEGRALS:

Ex: ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$



$$A = \int_{-a}^{a}\int_{y=-b\sqrt{1-(x/a)^2}}^{+b\sqrt{1+(x/a)^2}} dy\,dx \ldots \text{HARD}$$

instead, "transform" unit circle into the ellipse



$A = \int_{R_2} dA_2 = \int_{R_1} ab\,dA_1$

$= ab(\text{Area of unit circle})$

$= \pi ab$

[4/15/25] - lec 24 →
- vector fields
- div, grad, curl
- begin line ints

falstad.com/vector

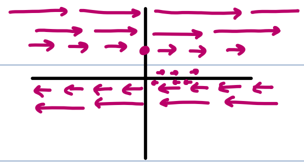
VECTOR FIELDS: function $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

for each (x,y), we visualize $\vec{F}(x,y)$ as vector coming out of that pt.

we've seen gradient fields e.g $f(x,y) = x^2+y^2 \to \vec{\nabla} f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

notation: $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$

Ex: $\vec{F}(x,y) = \begin{bmatrix} y \\ 0 \end{bmatrix}$



$\vec{F}(0,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{F}(x,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{F}(x,2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{F}(x,-1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

TWO CONCEPTS ASSOCIATED W/ vector fields:

① **DIVERGENCE** = "net inflow or outflow @ a point"
- connected to flux

② **CURL** = if you drop a flywheel in the field, will it spin?
- in which direction? how fast?

3D VECTOR FIELDS:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, same idea as 2D

notation: $\vec{F}(x,y,z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$

HOW TO COMPUTE DIVERGENCE & CURL?
- introduce diffrential operator

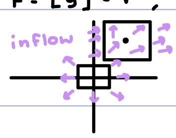
$\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$ or $\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$

"nabla"

we've already seen $\vec{\nabla} f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

given 3D $\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$, we can compute div $\vec{F} = \vec{\nabla} \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Ex: $\vec{F} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{r}$, $\operatorname{div}\vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y = 1 + 1 = \boxed{2}$



- Draw a box around pt.
- pts on sides of the box further from origin have longer vectors than pts closer to origin

net outflow & positive divergence

2D CURL: given $\vec{F}(x,y) = \begin{bmatrix} P \\ Q \end{bmatrix}$, 2D curl $\vec{F} = Q_x - P_y = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

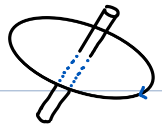
Ex: $\vec{F} = -\begin{bmatrix} y \\ x \end{bmatrix}$ → curl $\vec{F} = \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) = -2$

if you drop a flywheel in field, it'll spin CW b/c negative curl.

3D CURL: $\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$, curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{\imath} - (R_x - P_z)\hat{\jmath} + (Q_x - P_y)\hat{k}$$

INTERPRETATION of curl $\vec{F}$:



DIRECTION: direction in which you hold the stick of the flywheel

flywheel s.t. spins fastest CCW

MAGNITUDE: fastest spin rate

[4/16/25] - recitation

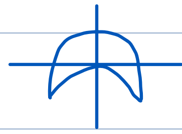
in one variable: start w/ x; then switch to $u(x) \rightsquigarrow du = u'(x)dx = \frac{\partial u}{\partial x} dx$

more than 1 var: start w/ x, y, then switch to $u = u(x,y)$, $v = v(x,y) \rightsquigarrow dudv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy = \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|$

Ex: calculate area of region bounded by $y^2 + 9x^4 + 6x^2y + x^2 = 1$
(HINT: find coords u & v s.t. R in coords is unit circle)

take $u = 3x^2 + y$ & $v = x$. then $\int_R dxdy = \int_{\text{unit circ.}} 1 \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} dudv$

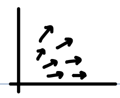
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} 6x & 1 \\ 1 & 0 \end{pmatrix}\right| = 1$$



$$= \int_{\text{unit circ.}} 1 \, dxdy = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{2}\, d\theta = \boxed{\pi}$$

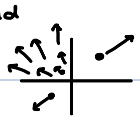
VECTOR FIELDS:

Exs: map of wind direction/magnitude



gravitational field

Ex: $\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath}$



"spout w/ water outwards"

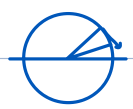
Ex: say we have large object of mass M @ (0,0). write the vector field corresponding to the gravitational force on a small mass m @ any given pt (x,y) in terms of $\vec{r}$ & G, M, m, $r = |\vec{r}|$ $\quad F = \frac{GMm}{r^2}$

$\vec{F}(x,y) = -x\hat{\imath} - y\hat{\jmath} \rightarrow$ always points inwards

$\frac{-\vec{r}}{r}$ is a vector field with unit length 1

so $\frac{GMm}{r^3}\vec{r}$ is vector field i want.

Ex: write down a vector field describing water flowing in circle CW @ constant speed
- circular motion @ const. speed



$\vec{p}(t) = (r\cos t, r\sin t)$

$\vec{v}(t) = (-r\sin t, r\cos t)$

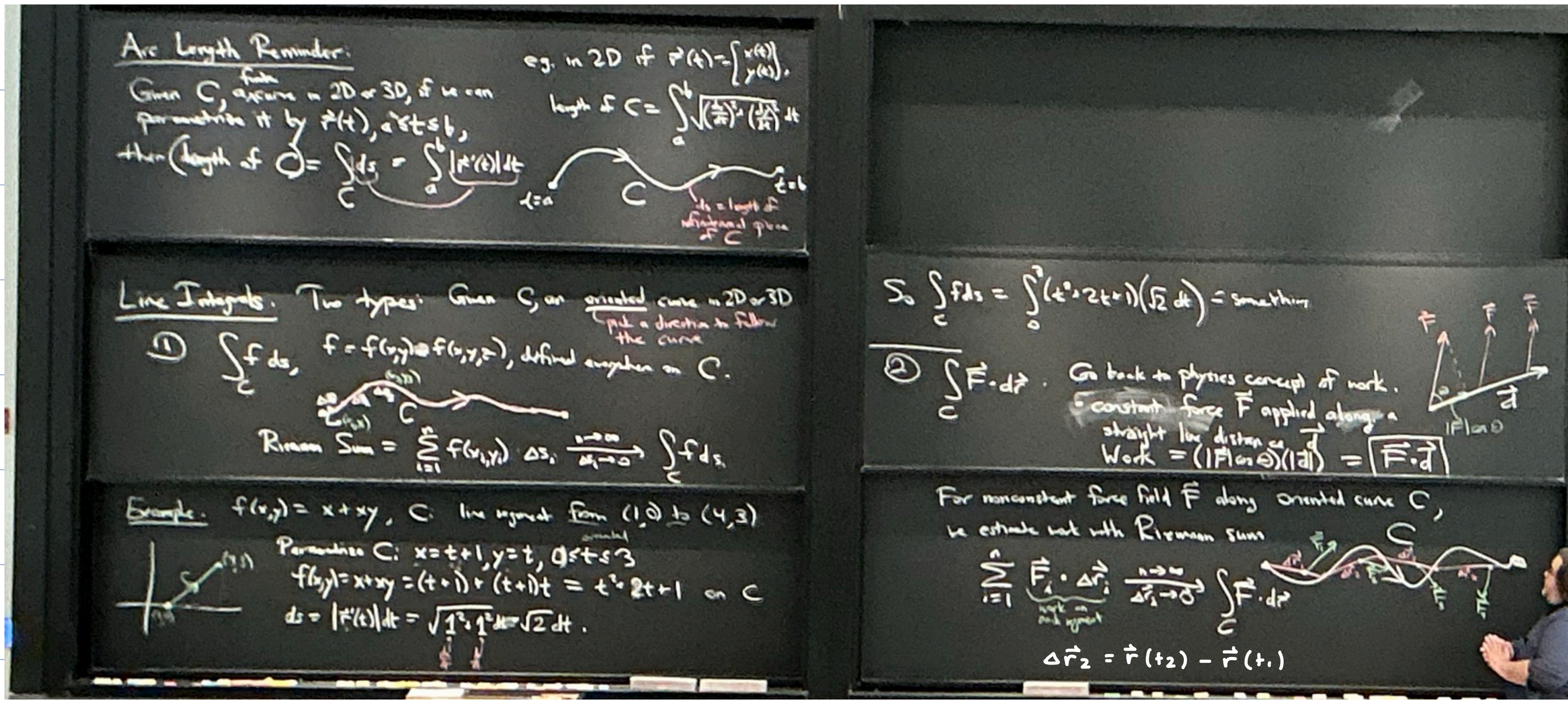
$\vec{p} \cdot \vec{v} = -r^2 \sin t\cos t + r^2 \sin t \cos t = 0 \leftarrow$ pos. & vel. $\perp$

@ each pt (x,y) we want $\vec{F}(x,y) \perp \vec{r}(x,y) = (x,y)$

how to choose a & b s.t. $ax + by = 0$? A: $a = -y$, $b = x$ or any multiple

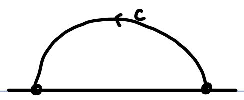
so $\vec{F}(x,y) = \underbrace{f(x,y)}_{\text{mystery}}(-y, x)$ $\qquad |\vec{v}(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$

[4/17/25] — · line ints vs. parameterization
· circulation



$d\vec{r} = \vec{r}\,'(t)\,dt$

EX: compute $\int_C \vec{F}\cdot d\vec{r}$



$\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\hat{\imath} + x\hat{\jmath}$

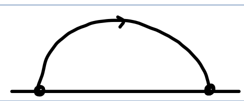
PARAMETERIZE C: $\vec{r}(t) = \begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix}$, $0 \le t \le \pi \rightarrow \vec{r}\,'(t) = \begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix}$

$\vec{F}(x,y) = F(\vec{r}(t)) = \vec{F}\left(\begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix}\right) = \begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix}$

so $\int_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{b} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = 9\pi$ ✓

$\begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix} \cdot \begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix} = 9$

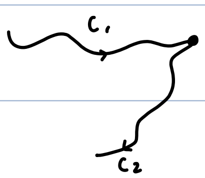
Same example, but CW:



$\int_C \vec{F}\cdot d\vec{r} = -9\pi$ b/c reverse all $(\vec{r}\,'(t))$ s

we often write $C_1 = -C$ → $\int_{-C} \vec{F} d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

orientation reversed

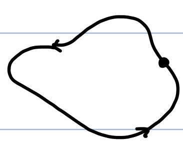


$C_1 + C_2$ = Follow C_1 then C_2

$\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

<u>CIRCULATION:</u> line integral around closed loop, oriented CCW

<u>NOTATION:</u> $\oint \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r}$



[4/22/25]-lecture → [· path-ind. = conservative = circulation-free = no curl
· green's thm.
· applications]

let $\vec{F}$ be 2D or 3D vector field.

· $\vec{F}$ is conservative iff $\vec{F}$ is gradient field
(i.e. $\vec{F} = \vec{\nabla} g$, g is called potential function for $\vec{F}$)

· $\vec{F}$ is path ind. iff line integrals $\int_C \vec{F}\cdot d\vec{r}$ depend only on starting & ending pt. not specific path.

· $\vec{F}$ is circulation free iff $\oint_C \vec{F}\cdot d\vec{r} = 0$ for closed loops C.

· given $\vec{F} = \langle P, Q \rangle$, 2D curl (scalar) is $Q_x - P_y$ (partial deriv) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\vec{F} = \langle P, Q, R \rangle$ 3D curl (vector) $\overrightarrow{\text{curl}}\ \vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \underbrace{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)}_{R_y - Q_z}\hat{\imath} + \underbrace{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)}_{P_z - R_x}\hat{\jmath} + \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{Q_x - P_y}\hat{k}$$

THM: if $\vec{F}$ is continuously differentiable, then the following are equivalent:
1) $\vec{F}$ is conservative
2) $\vec{F}$ is path-independent
3) $\vec{F}$ is circulation-free (integrate closed loop = 0)
4) curl $\vec{F} = 0$ (2D-scalar), curl $\vec{F} = \vec{0}$ (3D)

could prove all of these

Ex: path independence → circulation free

$\oint \vec{F}\cdot dA = \int_A^A \vec{F}\cdot d\vec{r} = 0$ (closed loop)
· can go backwards
another path that starts/ends @ A = don't go anywhere path

Ex: $\vec{F}$ is conservative → $\vec{F}$ is path independent
· this follows from fundamental thm. of Line Integrals
if $\vec{F} = \vec{\nabla} g$, then $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} g \cdot dr = g(B) - g(A)$

(doesn't depend on specific C)

Ex: conservative ⇒ curl $\vec{F} = \vec{0}$
say $\vec{F} = \langle P, Q, R \rangle$
if $\vec{F} = \vec{\nabla} g$, $g_x = P$, $g_y = Q$, $g_z = R$

$g_{xy} = P_y = g_{yx} = Q_x$ → z component of curl = 0
$g_{xz} = P_z = g_{zx} = R_x$ → y component of curl = 0
$g_{yz} = Q_z = g_{zy} = R_y$ → x component of curl = 0

Ex: curl $\vec{F} = \vec{0}$ → $\vec{F}$ is conservative

say $\vec{F} = \begin{array}{c} P \\ Q \\ R \end{array}\begin{bmatrix} y^2 + 3x^2 z + 1/x \\ 2xy + z + 2y \\ x^3 + y \end{bmatrix}\begin{array}{c} g_x \\ g_y \\ g_z \end{array}$ check "cross partials" are equal

$P_y = 2y = Q_x$ ✓
$P_z = 3x^2 = R_x$ ✓
$Q_z = 1 = R_y$ ✓

let's construct the potential function g. (via "partial anti diff.")

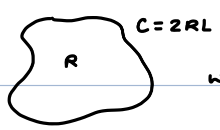
$g_x = P \to g = \int g_x\,dx = \int (y^2 + 3x^2 z + \frac{1}{x})\,dx = xy^2 + x^3 z + \ln x + C_1(y,z)$

$g_y = Q \to g = \int g_y\,dy = \int (2xy + z + 2y)\,dy = xy^2 + yz + y^2 + C_2(x,z)$

$g_z = R \to g = \int g_z\,dz = \int (x^3 + y)\,dz = x^3 z + yz + C_3(x,y)$

So, $\boxed{g(x,y,z) = xy^2 + x^3 z + yz + \ln x + y^2 + C}$

GREEN'S THM: (only in 2D) let R = finite region in xy plane

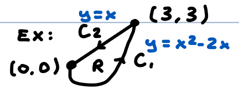


write $\overset{C}{\overset{\shortparallel}{\partial R}}$ for boundary of R, oriented CCW

let $\vec{F} = \langle P, Q \rangle$

line integral $\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{P\,dx + Q\,dy} = \iint_R \underbrace{(Q_x - P_y)}_{\text{curl } F \text{ (2D)}}\,dA$ double int equivalently, $\oint_C P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

or $\oint_C \vec F\cdot d\vec r = \int_R (\underbrace{\text{curl }\vec F}_{\text{2D curl}})\,dA$ (vector form)

Ex: 

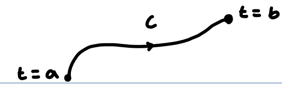
$\vec F = (x+y)\hat\imath + (3x+e^y)\hat\jmath$

notice: $\partial R = C_1 + C_2$

Green's: $\oint_{C_1+C_2} \vec F\cdot d\vec r = \int_R (\text{curl }\vec F)\,dA$ $Q_x - P_y = 3-1 = 2$

$= \int_R 2\,dA = 2(\text{Area of } R) = 2\int_0^3 \int_{x^2-2x}^{x} dy\,dx$

[4/23/25]-rec

LINE INTS: given curve C, parameterized by $\vec r(t)$, $a \le t \le b$, how to integrate 

$\int_C f\,ds$ ($ds = |r'(t)|\,dt$)

$= \int_a^b f(r(t))\,|r'(t)|\,dt$

other examples: $\int_C f\,dx = \int_C f(r(t))\,x'(t)\,dt$

$\int_C \vec F\cdot d\vec r = \int_a^b \vec F(\vec r(t))\cdot d\vec r(t)\,dt = \int_a^b F_1(\vec r(t))\cdot r_1'(t) + F_2(\vec r(t))\cdot r_2'(t)\,dt$

are these independent of the choice of path?

CONSERVATIVE VECTOR FIELD: $\vec F = \nabla \underset{\text{potential}}{f}$

(f = potential energy from gravity
$\vec F = \nabla f$ is gravitational field
b/c energy is conserved)

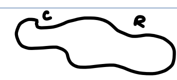
INDEPENDENT OF PATH:

THM: $\vec F$ is conservative $\Leftrightarrow$ $\int_C \vec F\cdot d\vec r$ is independent of path b/t two endpoints

curl$(\vec F) = 0$ $\Leftrightarrow$ $\oint_{\text{closed loops}} \vec F\cdot d\vec r = 0$ for any closed loop (circulation free)

4 equivalent conditions

*only need to find 1 of these to prove



GREEN'S THM: in 2D, if R is a region with no holes ("simply connected") and boundary C ccw

then $\oint_C \vec F\cdot d\vec r = \int_R \text{curl}(\vec F)\,dA$ ← in other words, $\oint_C F_1\,dx + F_2\,dy = \int_R \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\,dA$

EX: let C = circle $x^2+y^2=4$ oriented ccw. compute $\oint_C (2y+\sqrt{4+x^3})\,dx + (5x + e^{\tan^{-1} y})\,dy$

$\vec r(t) = (2\cos t, 2\sin t)$, $0 \le t \le 2\pi$

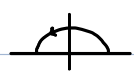
$d\vec r(t) = (-2\sin t, 2\cos t)$

in particular, $d\vec r$ @ a point (x,y) is just $(-y,x)$

then: want to find $\vec F$ s.t. $\oint_C \vec F\cdot d\vec r = \oint_C \vec F\cdot(-y,x) = \oint -F_1 y\,dx + F_2 x\,dy$

then for $\vec F = (2y+\sqrt{4+x^3}, 5x+e^{\tan^{-1}y})$, we get $\int_{\text{rad } 2} \text{curl}(\vec F)\,dA = \int -3\,dA = \int_0^{2\pi}\int_0^2 -3r\,dr\,d\theta = \int_0^{2\pi} -6\,d\theta = \boxed{-12\pi}$

curl$(\vec F) = \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 2-5 = -3$

EX: integrate $\vec F(x,y) = \langle x+y, 2x + y\ln(\csc\sqrt{1-y^5})\rangle$ over  half circle of radius 1

TRICK: $\int_{\frown} + \int_{\rightarrow} = \int_{\triangle} \vec F\cdot d\vec r = \int_{\triangle} \text{curl}(F)\,dA$ ← can apply Greens b/c closed curve

$\int_{\rightarrow} \vec F\cdot d\vec r = \int_{-1}^{1} \langle x, 2x\rangle\cdot\underbrace{\langle dx, 0\rangle}_{dr} = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0$

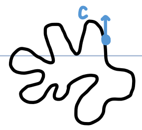
$\vec r(t) = (t,0)$, $-1 \le t \le 1$

$\int_{\Delta} \text{curl}(F)\, dA = \int_{\Delta} (1-2)\, dA = \int_{\Delta} -1\, dA = -\frac{\pi}{2}$

[4/24/24] - lecture 28 → $\left[\begin{array}{l}\cdot \text{greens thm} \\ \cdot \text{ param. surfaces / surface area}\end{array}\right.$

RECALL: $\vec{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$, $\vec{F}$ continuously differentiable on boundary of & inside plane region R

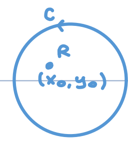
$C = \partial R$, oriented ccw (region "on the left")



then Green's thm says (vector form) $\oint_C \vec{F} \cdot d\vec{r} = \int_R \underbrace{(\text{curl } \vec{F})}_{\text{2D scalar curl}} dA$

(classical form) $\oint_C P\,dx + Q\,dy = \int_R (Q_x - P_y)\, dA$

tidbits: ① can prove immediately that $\text{curl } \vec{F} = 0 \Rightarrow \vec{F}$ is circulation free b/c $\oint_C \vec{F} \cdot d\vec{r} = \int_R (\text{curl } \vec{F})\, dA = \int_R 0\, dA = 0$



if R = tiny region containing (x_0, y_0) define circulation @ a point as $\text{circ}(x_0, y_0) = \lim_{(\text{area of } R \to 0)} \frac{\oint \vec{F} \cdot d\vec{r}}{\text{area of } R}$

② can justify idea that 2D curl measures "circulation @ a point."

if R is tiny, functions of R are approximately constant

in particular, for $(x,y) \in R$, $Q_x(x,y) \approx Q_x(x_0,y_0)$, $P_y(x,y) \approx P(x_0,y_0)$

$\rightarrow Q_x - P_y \approx$ constant

thus $\oint_C \vec{F} \cdot d\vec{r} = \int_R (Q_x - P_y)\, dA \approx (Q_x - P_y)(x_0,y_0) \cdot \int_R dA = (Q_x - P_y)(x_0,y_0)(\text{area of } R)$

so $\underbrace{\text{circ}(x_0,y_0) \approx (Q_x - P_y)(x_0,y_0)}_{\text{circ = curl } \vec{F}}$ with equality as (area of R) $\to 0$

③ you can often calculate areas of regions using Green's Thm

(see recitation 21 handout)

PARAMETERIZED SURFACES:

for parameterized curves, we had one parameter t, and

total derivative $D\vec{r}$ is 3×1 matrix $\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \vec{r}\,'(t) \leftarrow$ tangent to curve

$\vec{r}: \mathbb{R}^1 \to \mathbb{R}^3$

$t \longmapsto \vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

for parameterized surfaces, two params u, v: $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$

$\begin{bmatrix} u \\ v \end{bmatrix} \longmapsto \vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$

$D\vec{r}$ is 3×2 matrix $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = ?$

EX: plane spanned by $\vec{a} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -5\hat{\imath} + 7\hat{\jmath} + \hat{k} = \vec{n}$

eqn. of plane is $-5x + 7y + z = 0$

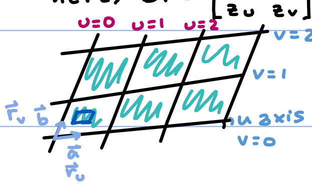
could parameterize as $\vec{r}_1(x,y) = \begin{bmatrix} x \\ y \\ 5x - 7y \end{bmatrix}$

or: take linear combos of a & b

$\vec{r}_2(u,v) = u\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + v\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2u + 3v \\ u + 2v \\ 3u + v \end{bmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$

check:
$-5(2u + 3v)$
$+7(u + 2v)$
$+ (3u + v)$ $= 0.$

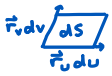
here, $D\vec{r} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$ get two tangent vectors to surface, $\vec{r}_u$ & $\vec{r}_v$



Q: what is surface area of plane for $0 \le u \le 3$, $0 \le v \le 2$

$|\vec{r}_u \times \vec{r}_v|$ = area of parallelogram

→ take a tiny parallelogram



dS = little piece of area of surface S.

$dS = |\vec{r}_u du \times \vec{r}_v dv| = |\vec{r}_u \times \vec{r}_v| du dv$

surface area $= \int_S dS = \int_{u=0}^{u=3} \int_{v=0}^{v=2} \underbrace{|\vec{r}_u \times \vec{r}_v|}_{|\langle -5,7,1\rangle| = 5\sqrt{3}} du dv = 6 \cdot 5\sqrt{3} = \boxed{30\sqrt{3}}$ (+ b/c flat)

[4/25/25] - lec

- surface int
- flux int
- $\vec{r}_u \times \vec{r}_v$ for various surfaces

SURFACE INTS: S surface in 3D param. by $\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$

last time: little pieces of area $= dS = |\vec{r}_u \times \vec{r}_v| du dv$

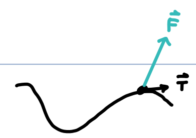
given $f(x,y,z)$ defined on S, we can compute $\int_S f(x,y,z) dS = \int_c^d \int_a^b f(x(u,v), y(u,v), z(u,v)) |\vec{r}_u \times \vec{r}_v| du dv$

back to line ints for a moment: C parameterized by $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ $a \le t \le b$

$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) |\vec{r}'(t)| dt$

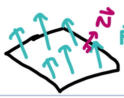
work $= \int_C (\text{comp. of } \vec{F} \text{ doing } C)\, ds = \int_C \vec{F} \cdot \vec{T}\, ds$

$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ $= \int \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| dt = \int_C \vec{F} \cdot d\vec{r}$ unit tan. vect.



FLUX: surface S, vector field $\vec{F}$ defined on & near S

$= \int_S (\text{component of } \vec{F} \text{ "}\underbrace{\text{through}}_{\perp}\text{" } S)\, dS$



flux of $\vec{F}$ through S $= \int_S \vec{F} \cdot \vec{N}\, dS$

NOTE: have to pick orientation of S, i.e. continuous choice of $\vec{N}$,
e.g. sphere centered @ (0,0,0) → outward orientation → $\vec{N}$ pointing away from origin
or
→ inward orientation

given param $\vec{r}(u,v)$ of S, $\vec{r}_u$, $\vec{r}_v$ tangent to S, so $\vec{r}_u \times \vec{r}_v$ is normal

then flux $= \int \vec{F} \cdot \vec{N}\, dS = \int_S \left(\vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right) |\vec{r}_u \times \vec{r}_v| du dv$

$\langle P, Q \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$

$= \int_{v=c}^{v=d} \int_{u=a}^{u=b} \vec{F}(\vec{r}(u,v)) \cdot \underbrace{(\vec{r}_u \times \vec{r}_v) du dv}_{d\vec{S}} = \underbrace{\int_S \vec{F} \cdot d\vec{S}}_{\text{flux}}$

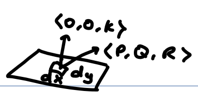
<u>NOTE:</u> $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \underbrace{\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}}_{\text{jacobian}} \hat{\imath} + \underbrace{\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}}_{\text{jacobian}} \hat{\jmath} + \ldots = \frac{\partial(y,z)}{\partial(u,v)} \hat{\imath} + \frac{\partial(z,x)}{\partial(u,v)} \hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)} \hat{k}$

$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$

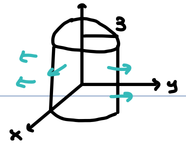
$\vec{F} \cdot d\vec{S} = \langle P, Q, R \rangle \cdot \left\langle \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right\rangle du dv$

in change of variables, $dx dy = \frac{\partial(x,y)}{\partial(u,v)} du dv$

$= \langle P, Q, R \rangle \cdot \langle dy dz, dz dx, dx dy \rangle$. flux $= \int_S P dy dz + Q dz dx + R dx dy$ (classical form)



Flux ex: cylinder $r=3$, $-5 \le z \le 5$



$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} = \begin{bmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{bmatrix} = \vec{r}(\theta, z)$, $0 \le \theta \le 2\pi$, $-5 \le z \le 5$

find flux of $\vec{F} = y\hat{\jmath}$ through S

top view:

$\vec{r}_\theta = \begin{bmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{bmatrix}$, $\vec{r}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{r}_\theta \times \vec{r}_z = (3\cos\theta)\hat{\imath} + (3\sin\theta)\hat{\jmath} + 0\hat{k}$

$\vec{F}(\vec{r}(\theta, z)) = 0\hat{\imath} + (3\sin\theta)\hat{\jmath} + 0\hat{k}$

flux $= \int_S \vec{F}\cdot d\vec{S} = \int_{z=-5}^{5}\int_{\theta=0}^{2\pi} \langle 0, 3\sin\theta, 0\rangle \cdot \langle 3\cos\theta, 3\sin\theta, 0\rangle \, d\theta\, dz = \int_{-5}^{5}\int_{0}^{2\pi} 9\sin^2\theta \, d\theta\, dz = \left(\int_{-5}^{5} dz\right)\left(\int_0^{2\pi} 9\left(\frac{1}{2}(1-\cos 2\theta)\right)d\theta\right)$

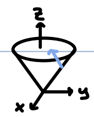
$= 10\left(9\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}\right) = \boxed{90\pi}$

[4/27/25] - lec. 30 $\left\{ \begin{array}{l} \cdot \text{ more flux exs} \\ \cdot \text{ flux shortcuts} \\ \cdot \text{ div. thm} \end{array} \right.$

- sphere
- cone

Flux ex:

CONE $z = \sqrt{x^2+y^2}$, $0 \le z \le 5$
oriented upwards



$\vec{F}(x,y,z) = \langle x, y, 0\rangle$ find flux of $\vec{F}$ through cone

flux $= \int_S \vec{F}\cdot d\vec{S} = \int \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$

parametrize curve: $z = r$ in cylindrical coords

$(r, \theta, z) \rightsquigarrow (r, \theta, r)$

so $\vec{r}(r,\theta) = \langle x, y, z\rangle = \langle r\cos\theta, r\sin\theta, \underset{\text{or } z}{r}\rangle$

$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle = r\langle -\cos\theta, \sin\theta, 1\rangle$ oriented up

$\vec{F}(\vec{r}(r,\theta)) = \langle r\cos\theta, r\sin\theta, 0\rangle$

so flux $= \int_{\text{core}} \vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta)\, dr\, d\theta = \int_0^{2\pi}\int_0^{5} r^2 (\underbrace{-\cos^2\theta - \sin^2\theta}_{-1})\, dr\, d\theta = -\int_0^{2\pi} d\theta \int_0^{5} r^2\, dr = -2\pi\left(\frac{1}{3}5^3\right) = \boxed{\frac{-250\pi}{3}}$

SPHERE FLUX EXAMPLE: sphere of radius 7, center @ origin, oriented out

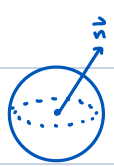
$\vec{F}\langle x, y, z\rangle$. Find flux.

sphere: $\rho = 7$

param: $r(\phi, \theta) = \begin{bmatrix} 7\sin\phi\cos\theta \\ 7\sin\phi\sin\theta \\ 7\cos\phi \end{bmatrix}$

REMINDER: $x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7\cos\phi\cos\theta & 7\cos\phi\sin\theta & -7\sin\phi \\ -7\sin\phi\sin\theta & 7\sin\phi\cos\theta & 0 \end{vmatrix} = 49\sin\phi \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}$



$\vec{F}(\vec{r}(\phi,\theta)) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7\sin\phi\cos\theta \\ 7\sin\phi\sin\theta \\ 7\cos\phi \end{bmatrix}$

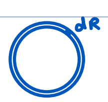
METHOD 1: Plug & Chug

$\int_{\text{sphere}} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} 7^3 \sin\phi\, (\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi)\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} 7^3 \sin\phi\, d\phi\, d\theta = 7^3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right) = 4\pi 7$
$= 7\ (\text{SA})$
$= 3\ (\text{vol})$

vol. of sphere of rad R: $\frac{4}{3}\pi R^3 = \frac{d}{dR}\left(\frac{4}{3}\pi R^3\right)$

SA: $\frac{d}{dA}(\pi r^2) = 2\pi r$



Method #2:

use fact that $\vec{F}$ & $\hat{n}$ are related use flux $= \int_S \vec{F}\cdot d\vec{S} = \int (\vec{F}\cdot\hat{n})\,dS$

in this example, $\vec{F}$ & $\hat{n}$ are parallel.

so $\vec{F}\cdot\hat{n} = |\vec{F}|\cdot|\cancel{\hat{n}}|\cancel{\cos\theta} = |\vec{F}| = \sqrt{x^2+y^2+z^2} = \rho = 7$

so flux $= \int_{sph} (\vec{F}\cdot\hat{n})\,dS = \int_S 7\,dS = 7\underbrace{\left(\int dS\right)}_{SA} = 7(SA) = 7(4\pi 7^2)$ ✓

ASIDE: similarly, if $\vec{F}$ is always $\perp$ to surface, then $\vec{F}\cdot\hat{n} = 0$, so flux $= 0$
e.g. S = sphere, $\vec{F} = \langle -y, x, 0\rangle$

METHOD 3: DIVERGENCE THM

REMINDER: given field $\vec{F} = \langle P, Q, R\rangle$, $\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

THM: let S = closed surface, oriented outward.
let $\vec{F}$ be field which is continuously diffrentiable on & inside S

$S = \partial T$ "boundary of T", T = space region inside S

then $\oint_{S=\partial T} \vec{F}\cdot d\vec{S} = \int_T (\operatorname{div}\vec{F})\,dV$

when surface is closed

EX: S = sphere rad 7 oriented out
$\vec{F} = \langle x, y, z\rangle \to \operatorname{div}\vec{F} = 3$

so Divergence Thm $\Rightarrow$ flux $= \oint_{sph.} \vec{F}\cdot d\vec{S} = \int_{inside} 3\,dV = 3(\text{vol.}) = 3\left(\frac{4}{3}\pi 7^3\right)$

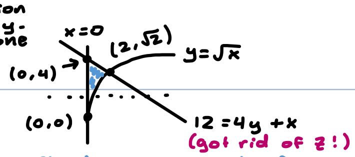
[4/30/25] - OH - test prep

③ $y = \sqrt{x}$ $\quad \int_{\substack{\text{region}\\ \text{in } xy\text{-plane}}}$ upper bound for z over (x,y) - lower bound for z over (x,y) $dx\,dy$

$x = 0$

$z = 2x + y$

$x + 12 = 3y + z$



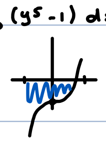
$x + 12 = 3y + 2x + y \to 12 = 4y + x$

$12 = 4\sqrt{x} + x$

in terms of x: $\int_0^4 \int_{\sqrt{x}}^{\frac{12-x}{4}} \int_{\ldots}$

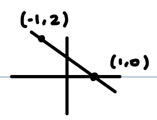
in terms of y: would have to break into 2 integrals!

⑤ a) $\int_D (y^5 - 1)\,dx\,dy$ $\quad$ bounded by $x^2 + y^2 \le 1$



b) $\int \cos\phi\,dV$ ✓ $\quad \phi$ zero → pos ... $\boxed{\text{pos}}$

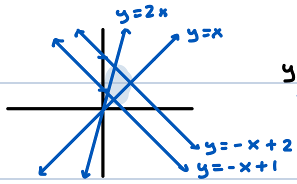
④ c) $\int x\,dy$ $\quad$ param: $\int_{-1}^{1} x(t)\cdot c\,dt = \int_{-1}^{1} t\cdot b\,dt$ $\quad \boxed{\text{zero!}}$



$\vec{r}(t) = (t, a + bt)$
$-1 \le t \le 1$

- change of y is same
- each segment will be cancel
- if $\int y\,dx$, would be $\ominus$ b/c height matters
- if func. shifted, would change $\int x\,dy$

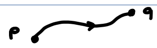
⑦ $y=x$ $y=2x$ $y=-x+1$ $y=-x+2$ $u=x+y$ $v=\frac{y}{x}$



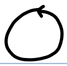
$y=x$	$y=2x$	$y=-x+1$	$y=-x+2$
$v=1$	$v=2$	$u=1$	$u=2$

$\int_1^2 \int_1^2$ make func. in terms of u & v $du\,dv$

⑧ b)



$\int_c \vec{\nabla} f \cdot d\vec{r} = f(q) - f(p)$


← circ free $\oint_c \vec{\nabla} f \cdot d\vec{r} = 0$

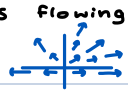
$\vec{F}$ is $\vec{\nabla} f$ ("conservative") = circulation free

[4/30/25] - rec - test prep

VECTOR FIELDS: at each pt (x,y), we get a vector $\vec{F}(x,y) = (F_1(x,y), F_2(x,y), F_3(x,y))$

think: how water is flowing @ pt (x,y)

Ex: $\vec{r}(x,y) = (x,y)$



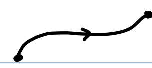
$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

DERIVATIVES OF FUNCS / VECT. FIELDS:

- GRADIENT: if $f(x,y,z)$ is function, then gradient is $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ (functions → vector fields)
- CURL: 2D: if $\vec{F}$ is vector field, $\text{curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ (2D vector field → function)

  3D: takes vector field $\vec{F} = (F_1, F_2, F_3)$ ⇝ vector field $\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \text{etc.}\right)$ $\det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$
- DIVERGENCE: takes 3D vector field $\vec{F} = (F_1, F_2, F_3)$ to $\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

LINE INTEGRALS: $\int_c f\,ds$ means: parametrize C by $\underset{(x(t),\,y(t))}{\vec{r}(t)}$, $a \le t \le b$, then $\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$



$\vec{r}\,'(t) = (x'(t), y'(t))$

$|\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

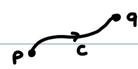
$\int_c f\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$

$\int_c \vec{F} \cdot d\vec{r} = \int_a^b F_1\,dx + F_2\,dy + F_3\,dz = \int_a^b F_1(r(t))x'(t) + F_2(r(t))y'(t) + F_3(r(t))z'(t)\,dt$

an especially nice line int: $\int_c \vec{F} \cdot d\vec{r}$ when $\vec{F} = \vec{\nabla} f$ for some function f

("$\vec{F}$ is conservative" & "f is the potential function")

FUNDAMENTAL THM OF LINE INTEGRALS: $\int_c \underbrace{\vec{\nabla} f}_{\vec{F}} \cdot d\vec{r} = f(q) - f(p)$



how do we know if $\vec{F}$ is conservative? if so, how to find f?

- $\text{curl}(\vec{F}) = 0$
- path independence: if C & C' both paths from p to q, then $\int_c \vec{F} \cdot d\vec{r} = \int_{c'} \vec{F} \cdot d\vec{r}$
- circulation free: if C is closed loop, then $\oint_c \vec{F} \cdot d\vec{r} = 0$

if $\vec{F}$ is conservative, how to find f s.t. $\vec{F} = \vec{\nabla} f$, i.e. $(F_1, F_2, F_3) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$f = \int F_1\,dx + C(y,z)$ + some function $g_1(y,z)$

$f = \int F_2\,dy + C(x,z)$ + some function $g_2(x,z)$

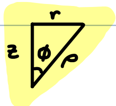
$f = \int F_3\,dz + C(x,y)$ + some function $g_2(x,y)$

b/c for each fixed y_0, z_0, $F_1(x_0, y_0, z_0)$ & $f(x_0, y_0, z_0)$ are now functions of one variable, so $f(x_0, y_0, z_0) = \int F_1(x_0, y_0, z_0) + C$ for some C

CENTER OF MASS: given region R with some density ρ. find CoM.
find $x, y, \& z$ coords. seperately
the x-coordinate of CoM is weighted avg. of x-coords of pts in region

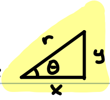
$$= \frac{\int_R x\rho \, dV}{\int_R \rho \, dV}$$

CYLINDRICAL → SPHERICAL:



AREAS:
rect: $dx\,dy\,dz$
cylin: $r\,dr\,d\theta\,dz$
spher: $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

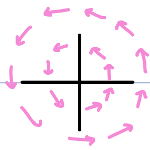
RECT → POLAR:



[5/6/25] - lecture →
- extended Greens Thm
  - decaying rot. vect. fields
- extended div thm.
  - decaying rad. vect. fields

back to 2D greens thm for a bit:

Ex: $\vec{F} = \langle -y, x \rangle$
integrate around
$C: x^2+y^2=R^2$ ccw



1) direct way: $\oint_C \vec{F}\cdot d\vec{r}$ $\quad \vec{r}(t) = \langle R\cos t, R\sin t \rangle, \quad 0 \le t \le 2\pi$

$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} R^2 \, dt = 2\pi R^2$ $\quad \vec{r}\,'(t) = \langle -R\sin t, R\cos t \rangle$

2) greens: $\oint \vec{F}\cdot d\vec{r} = \int_{\text{ins.}} (\text{curl } \vec{F}) \, dA = \int_{\text{in}} (Q_x - P_y) \, dA = \int 2 \, dA = 2\pi r^2$

Similar field: $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \frac{1}{r^2} \langle -y, x \rangle, \quad r = \sqrt{x^2+y^2}$

$|\vec{F}| = \frac{1}{r^2} |\langle -y, x \rangle| = \frac{1}{r^2} r = \frac{1}{r}$ ← vectors get shorter as you go away from origin

integrate around $C: x^2+y^2=R^2$ $\quad$ same $\vec{r}, \vec{r}\,'$

$\vec{F}(\vec{r}(t)) = \left\langle \frac{-y}{R^2}, \frac{x}{R^2} \right\rangle = \left\langle -\frac{\sin t}{R}, \frac{\cos t}{R} \right\rangle$

then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left(\frac{-\sin t}{r}\right)(-R\sin t) + \left(\frac{\cos t}{R}\right)(R\cos t) = 1$

So, $\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 1 \, dt = 2\pi$, independent of R.

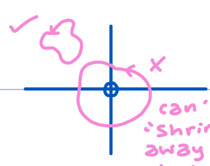
2) greens: $Q_x - P_y \rightarrow Q_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$P_y = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore Q_x - P_y = 0$. So, Greens Thm says $\oint \vec{F}\cdot d\vec{r} = \int_{\text{in}} 0 = 0$, NOT 2π → we got different answers

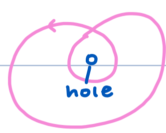
★ you can only apply Greens Thm to continuously differentiable curves / field — this is und. @ origin! & inside C contains $(0,0)$

domain of $\vec{F} = \mathbb{R}^2 - \{(0,0)\}$



can't "shrink" away the hole

NOT simply connected
any closed curve can be shrunk to nothing



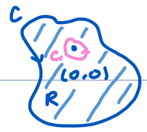
4π
can find $n(2\pi)$ where n = winding #

to use green's thm, region inside C and on which $\vec{F}$ is defined must be simply connected.

non self intersecting

CLAIM: for $\vec{F} = \left\langle \frac{-y}{r^2}, \frac{x}{r^2} \right\rangle$, if C is any simple loop around (0,0), oriented CCW, then $\oint \vec{F}\cdot d\vec{r} = 2\pi$.

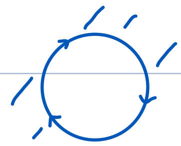
why?



draw circle centered @ (0,0), fully inside C.

we know $\int_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$

let R = region outside C_1, inside C



1) ∂R is made up of C & C_1

in fact, RHR says we should orient C ccw but C_1 cw (when considered as boundary of R)

write $\partial R = C - C_1$

b/c R does not contain (0,0), $\vec{F}$ is defined everywhere so Green's Thm. applies to $(R, \vec{F})$

$$\oint_{\partial R} \vec{F}\cdot d\vec{r} = \int_R \underbrace{(Q_x - P_y)}_{0} dA = 0$$

$$\oint_{C=C_1} \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} - \underbrace{\oint_{C_1} \vec{F}\cdot d\vec{r}}_{2\pi} = 0$$

so $\boxed{\oint_C \vec{F}\cdot d\vec{r} = 2\pi}$

more general $\vec{F} = \frac{1}{r^n}\langle -y, x\rangle$

- $n=0$: $\vec{F}_0 = \langle -y, x\rangle$ curl $\vec{F}_0 \neq 0$
- $n=2$: $\vec{F}_2 = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ curl $\vec{F}_2 = \vec{0}$

what is $Q_x - P_y$ for this in general? $\boxed{Q_x - P_y = \frac{2-n}{r^n}}$

standard radial

2D flux: $\vec{F} = \frac{\langle x, y\rangle}{r^n}$, div $\vec{F} = P_x + Q_y \rightarrow P_x + Q_y = \frac{2-n}{r^n}$

3D flux: $\vec{F} = \frac{\langle x, y, z\rangle}{\rho^n} \Rightarrow \text{div}\,\vec{F} = \frac{3-n}{\rho^n}$ (1)

div $\vec{F} = 0$ when $n = 3$ $\quad |\vec{F}| = \frac{\rho}{\rho^n} = \frac{1}{\rho^{n-1}}$

$\hookrightarrow$ div $\vec{F} = 0$ when $|\vec{F}| = \frac{1}{\rho^2}$

[5/8/25] - lecture 33 →
- Greens Thm
- "compatible" orientations via RHR
- Stokes thm

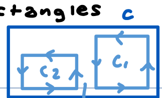
outline of proof of green's thm: (2D) region R, $\partial R = C$, closed, oriented ccw

$\vec{F}(x,y) = \langle P, Q\rangle$ is continuously differentiable on R & C

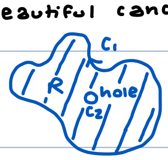
then $\oint_R \vec{F}\cdot d\vec{r} = \int_R (\text{curl}\,\vec{F})\, dA$ 2D curl $Q_x - P_y$

TWO-STEP PROOF:

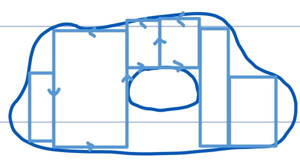
1) show directly Green's Thm holds for rectangles
2) beautiful cancellation



cancel!



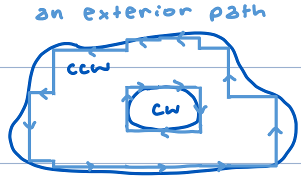
→



$$\int_{C_1} + \int_{C_2} \vec{F} dr$$

*all the interior paths cancel, and we get an exterior path

$$= \int_C \vec{F}\cdot d\vec{r}$$



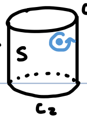
- approximate R w/ rectangles
- circulation along shared (interior)
- edges cancel
- what's left is boundary

moreover, if there's a hole in R, from rectangle approx., interior is CW oriented

$\partial R = C_1 - C_2$
CCW CW

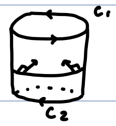
SURFACES W/ BOUNDARY:

Ex: cylinder, finite, open @ top & bottom



boundary S is made up of circles C_1 & C_2
let's say C_1 oriented as shown.
an orientation on a boundary curve induces an orientation on the surface, vice versa. ← via RHR

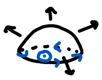
given C_1's orientation, pick a pt on S near C_1, curl your fingers (on right hand) so that they follow C_1, then whichever way thumb is pointing is induced orientation on S.
in this example, thumb points into cylinder, so S oriented inwards



pick pt on S near C_2,
thumb points inward, so fingers curl along C_2 as shown

fingers follow C_1. whichever way thumb pts is induced orientation on S. thumb pts into cyl, so S oriented in!

Ex 2: Hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, oriented inwards



oriented in
pt near equator, thumb pts out,
fingers curl, induced orientation ∂S

STOKES THM: (3D version of Green's)

S a surface with boundary with an orientation, ∂S = curve(s) with <u>compatible</u> orientation, $\vec{F} = \langle P, Q, R \rangle$

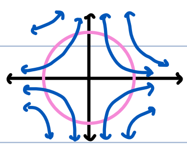
then $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_S (\overrightarrow{\text{curl}}\ \vec{F}) \cdot d\vec{S}$
line int. surface int.

reminders: $\overrightarrow{\text{curl}}^{3D}\ \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

green's thm. as special case of Stokes: if S is contained in xy-plane, $\vec{F} = \langle P, Q, 0 \rangle$ (so $S = R$)
oriented up, $\hat{n} = \langle 0, 0, 1 \rangle$ curl $\vec{F} \cdot \hat{n} = Q_x - P_y$

$$\oint \vec{F} \cdot d\vec{r} = \int_R (Q_x - P_y)\, dA$$

Ex: $\vec{F}(x,y,z) = \langle xz - y, x - yz, x^2 - y^2 + 5z \rangle$
S = part of surface $z = xy$ whose shadow in xy-plane is circle of rad = 3.



∂S = space curve in surface $z = xy$ whose shadow in xy plane is circle of $r=3$

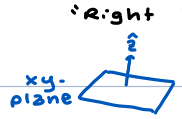
compute $\oint_{\partial S} \vec{F} \cdot d\vec{r}$

[5/12/25]-rec

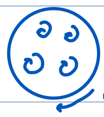
GENERALIZED STOKES THM: $\int_{\partial R} \omega = \int_R d\omega$

	2D $\vec{F} = (F_1, F_2)$	3D $\vec{F} = (F_1, F_2, F_3)$
dim R = 1 (a, R, b)	Fundamental theorem of line integrals (func. 2 var) $f(b)-f(a) = \int_R \nabla f \cdot d\vec{r}$ ($\nabla f = \vec{F}$)	Fundamental theorem of line integrals (f = func 3 vars) $f(b)-f(a) = \int_R \nabla f \cdot d\vec{r}$ ($\nabla f = \vec{F}$)
dim R = 2	green's thm $\oint_{\partial R} \vec{F}\cdot d\vec{r} = \int_R \text{curl}(\vec{F})\, dA$; $\text{curl}(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$	Stoke's thm: if R is oriented on surface in 3D w/ oriented boundary R, then $\oint_{\partial R} \vec{F}\cdot d\vec{r} = \int_R (\vec{\nabla}\times\vec{F})\cdot d\vec{R}$
dim R = 3	×	divergence thm: if R is a 3D region & ∂R is oriented outwards, $\oint_{\partial R} \vec{F}\cdot d\vec{r} = \int_R (\vec{\nabla}\cdot\vec{F})\, dV$

STOKES THM: what does it mean for a surface S & boundary ∂S to have compatible orientations?
"Right hand rule"



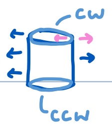
RHR: thumb in $\hat{z}$ dir.
fingers: ccw



oriented inward

is the matching orientation of boundary

Ex:



choices: in or out
we're choosing outwards

orient out, cw

MORALE: point thumb in direction the surface is oriented
curl fingers to indicate current direction the boundary should be oriented in

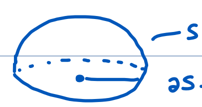
Ex: let S = upper half of sphere radius = 2 oriented upwards
let $G = \langle x, 0, z\rangle$. calculate the flux of G through S.

$\int_S \vec{G}\cdot d\vec{S}$

hint: show $\vec{G} = \vec{\nabla}\times\vec{F}$ for $\vec{F} = \langle 0, xz, 0\rangle$

check: $\vec{\nabla}\times\vec{F} = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xz & 0 \end{pmatrix} = \hat{\imath}\left(\frac{\partial 0}{\partial y} - \frac{\partial xz}{\partial z}\right) - \hat{\jmath}(0) + \hat{k}\left(\frac{\partial xz}{\partial x} - 0\right) = \begin{pmatrix} -x \\ 0 \\ z \end{pmatrix}$

trying to calculate $\int_S \vec{G}\cdot d\vec{S} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} \overset{\text{Stokes}}{=} \oint_{\partial S} \vec{F}\cdot d\vec{r}$



∂S = circle $x^2+y^2=4$ $(z=0)$

$\vec{F} = \langle 0, xz, 0\rangle$

but on ∂S we have $z=0$
so in fact $\vec{F} = 0$
so $\oint_{\partial S} \vec{F}\cdot d\vec{r} = 0$

[5/13/25] - lecture 35

- fund. thm of calc.
- differential forms
- generalized stokes

FUNDAMENTAL THMS OF CALC:

OG: $\int_a^b f'(x)\,dx = f(b) - f(a)$

LINE INT: $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$ or $\int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f(B) - f(A)$

(vector form) (classical differential form)

GREEN'S THM: (2D) $\int_R \text{curl}\,\vec{F}\; dA = \oint_{\partial R} \vec{F}\cdot d\vec{r}$ or $\underbrace{\int_R (Q_x - P_y)\,dx\,dy}_{\text{2 form}} = \oint_{\partial R} \underbrace{\overset{w}{P\,dx + Q\,dy}}_{\text{1 form}}$

STOKE'S THM: (3D) $\int_S (\text{curl}\,\vec{F})\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{S}$ or $\int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$
$= \oint P\,dx + Q\,dy + R\,dz$

DIVERGENCE THM: $\int_T (\text{div}\,\vec{F})\,dV = \oint_{\partial T} \vec{F}\cdot d\vec{S}$ or $\int_T \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz}_{\text{3 form}} = \oint_T \underbrace{P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy}_{\text{2 form} \quad w}$

space region T,
$\partial T = S$

DIFFERENTIAL FORMS: "things we can integrate"

- 0-form: function $f(x,y)$ or $f(x,y,z)$
- 1-form: $f(x)\,dx$ or $P\,dx + Q\,dy + R\,dz$
- 2-form: $f(x,y)\,dx\,dy$ or $P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$
- 3-form: $f(x,y,z)\,dx\,dy\,dz$

RULES ab. differential forms:

1) $dy\,dx = -dx\,dy$ $\qquad \frac{\partial(y,x)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$d\vec{x} \times d\vec{y} = -(d\vec{y} \times d\vec{x})$

$\Rightarrow dx\,dx = -dx\,dy$
$\Rightarrow dx\,dx = 0$

2) $f(x,y,z)$: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ $\qquad d: \{0 \text{ forms}\} \to \{1 \text{ forms}\}$

3) $d(dx) = 0$ taking a tiny piece of a tiny piece = ignore!

4) $d(f+g) = df + dg$, $d(fg) = f\,dg + g\,df$ (sum of product rule)

(extended) $d(w + \alpha) = dw + d\alpha$, $d(w\alpha) = w\,d\alpha + \alpha\,dw$

ASIDE: say $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$
$= \cos\theta\,dr - r\sin\theta\,d\theta$
$dy = \sin\theta\,dr + r\cos\theta\,d\theta$

then $dx\,dy = (\cos\theta\,dr - r\sin\theta\,d\theta)(\sin\theta\,dr + r\cos\theta\,d\theta)$

$= (\text{blah})\,dr\,dr + r\cos^2\theta\,d\theta\,dr$
$+ (\text{blah}_2)\,d\theta\,d\theta$
$= r\cos^2\theta\,dr\,d\theta + r\sin^2\theta\,dr\,d\theta$

$dx\,dy = r\,dr\,d\theta$

if $w = P\,dx + Q\,dy$ (no dz's),
then $dw = d(P\,dx) + d(Q\,dy)$

$= P\cancel{d\,dx} + dP\,dx + Q\cancel{d\,dy} + dQ\,dy$

$\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)dy$

$= \frac{\partial P}{\partial y}dy\,dx + \frac{\partial Q}{\partial x}dx\,dy$

$= -\frac{\partial P}{\partial y}dx\,dy + \frac{\partial Q}{\partial x}dx\,dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

now Green's says: if $w = P\,dx + Q\,dy$ is 1-form in 2D space, & if R is 2D region w/ boundary ∂R,

then $\int_R dw = \int_{\partial R} w$ — 1 form

2D, 2 form, 1D

if $w = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$

then $dw = d(P\,dy\,dz) + d(Q\,dz\,dx) + d(R\,dx\,dy)$

$$= dP\,dy\,dz + dQ\,dz\,dx + dR\,dx\,dy$$

$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)dy\,dz + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right)dz\,dx + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right)dx\,dy$$

$$= \frac{\partial P}{\partial x}dx\,dy\,dz + \frac{\partial Q}{\partial y}\underbrace{dy\,dz\,dx}_{=+dx\,dy\,dz} + \frac{\partial R}{\partial z}\underbrace{dz\,dx\,dy}_{=+dx\,dy\,dz}$$

$$= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

then Div Thm says: 2 form w in 3D space.

3D region T w/ 2D boundary ∂T

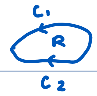
$\boxed{\text{then } \int_T dw = \int_{\partial T} w}$ generalized Stoke's Thm

"move the d & change font"

plug in pts "0-integral"

FUNDAMENTAL THM OF LINE INTS: $\int_C \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}_{df} = f\Big|_A^B = f\Big|_{\partial C}$

$\partial C = B - A$



$\partial R = C_1 - C_2$

$\int_C df = \Big|_{\partial C} f$