Cattu - 18.02 final

IDENTITY: $\begin{bmatrix}1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\cdots\end{bmatrix}$
→ doesn't change mat.

DET: $\det\begin{bmatrix}a&b\\c&d\end{bmatrix} = ad-bc$

$\det\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = a(ei-gh) - b(di-fg) + c(dh-eg)$

↑ 2x2 area squaring fact.
3x3 vol. squaring

DOT PRODUCT: (scalar) $\vec{v}\cdot\vec{w} = |\vec{v}|\,|\vec{w}|\cos\theta$, $\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right)$
if ⊥: $\vec{v}\cdot\vec{w}=0$

CROSS PRODUCT: (vector) $\begin{bmatrix}a\\b\\c\end{bmatrix}\times\begin{bmatrix}d\\e\\f\end{bmatrix} = \langle bf-ec, -(af-dc), ae-bd\rangle$
(-) = opposite dir. if ∥: $\vec{v}\times\vec{w}=0$

AREA: △ triangle area $= \frac{1}{2}\|\vec{v}\times\vec{w}\|$

COMPONENT: (scalar) $comp_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$ or $comp_{\hat{w}}\vec{v} = (\vec{v}\cdot\hat{w})$

PROJECTION: (vector) $proj_{\vec{w}}\vec{v}\left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$ or $proj_{\hat{w}}\vec{v} = (\vec{v}\cdot\hat{w})\hat{w}$
proj. of $\vec{v}$ onto $\hat{w}$

INVERSE: $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ ← if det ≠ 0 $\vec{x} = A^{-1}\vec{b}$

PLANE: pt (x,y,z) & normal vector $\vec{n} = \langle a,b,c\rangle \to ax+by+cz=d$

$\vec{n} = \vec{AB}\times\vec{BC} = \langle a,b,c\rangle$
pt: $A = (x_0,y_0,z_0)$
$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$
$d = ax_0+by_0+cz_0 \to ax+by+cz=d$

SHORTEST DIST: $d = \frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}$ pt: (x_1,y_1,z_1) & $n=(a,b,c)$

MATRICES: associative $(AB)C = A(BC)$, distributive $A(B+C) = AB+AC$

ROW RED: $\left[\begin{array}{cc|c}2&3&5\\1&-1&6\end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c}1&4&-1\\1&-1&6\end{array}\right]$ add/sub. swap multiply

SPAN: set of all linear combos $\vec{v}_1,\ldots,\vec{v}_k$
Ex: $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}3\\4\end{bmatrix}$ → span$\{v_1,v_2\} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}c_1+3c_2\\2c_1+4c_2\end{bmatrix}$

INDEPENDENT: if $c_1v_1+c_2v_2+\ldots=0$, **DEPENDENT** if $c_1v_1+\ldots\neq 0$
→ use row reduction to determine; if whole row = 0 → depend!

MATRIX PROD: $\underbrace{a\times b * b}_{\text{must} =}\times c$ → if det = 0 → depend!
· non-communicative

ARC LEN: $L = \int_a^b\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}$

if det = 0:
a) span & linear dependence of cols
b) matrix not invertible
c) no unique soln → no soln. or ∞

if det ≠ 0:
a) span & linear independence of cols
b) matrix IS invertible
c) unique soln → $\vec{x} = A^{-1}b$

EIGENVALS/EIGENVECTS: $A\vec{v} = \lambda\vec{v}$, $\det(A-\lambda I)$

DIAGONALIZATION: $A = PDP^{-1}$, $P^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$D = \begin{bmatrix}\lambda&0&\cdots\\0&\lambda&\cdots\\0&\cdots&\lambda\end{bmatrix}$, $P = \begin{bmatrix}v_{11}&v_{21}\\v_{12}&v_{22}\end{bmatrix}$ eigenvect.

COMPLEX #s: $re^{i\theta} = r(\cos\theta + i\sin\theta)$, $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

ROOTS: $z_k = \sqrt[n]{r}\,e^{\frac{i(\theta+2\pi k)}{n}}$

POWERS: $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

DIRECTIONAL DERIV: $D_{\hat{u}}f = \nabla f\cdot\hat{u} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$
max dd $= |(\nabla f)(P)|$ u = unit vector!

Ex: $D_{\hat{u}}f$ @ (1,2) for $f(x,y) = x^2y + e^y$ in dir. $\vec{u} = \frac{1}{\sqrt{2}}(1,1)$ $\nabla f = (2xy, x^2+e^y)$
$\nabla f(1,2) = (4, 1+e^2)$
$D_{\hat{u}}f = \frac{4}{\sqrt{2}} + \frac{1+e^2}{\sqrt{2}}$

LINEAR APPROX: $f(x,y) = f(a,b) + f_x(x-a) + f_y(y-b)$

TAN PLANE: $z - f(a,b) = \underbrace{(f_x(a,b))}_{\#}(x-a) + \underbrace{(f_y(a,b))}_{\#}(y-b)$
$(\vec{\nabla}f)(P)$ = ⊥ to tan plane @ P

GRADIENT: $\vec{\nabla}f = \left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$ ← points in dir. of steepest ascent

Ex: approximate $f(x,y) = x^2+y^2$ near (1,2)
$f_x = 2x \to f_x(1,2) = 2$ (1.1, 2.05)
$f_y = 2y \to f_y(1,2) = 4$
$f(1.1, 2.05) = (1^2+2^2) + 2(0.1) + 4(0.05)$
$df = 2x\,dx + 2y\,dy$
$= 2(1)(0.1) + 2(2)(0.5)$

JACOBIAN: $D_f = \begin{bmatrix}\frac{\partial z}{\partial x}&\frac{\partial z}{\partial y}\\\frac{\partial w}{\partial x}&\frac{\partial w}{\partial y}\end{bmatrix}$

MULTIVAR CHAIN: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

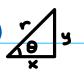
CRITICAL PTS: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

2ND DERIV. TEST: $D = \overbrace{f_{xx}\cdot f_{yy} - (f_{xy})^2}^{\text{Hessian}}$
D>0 & $f_{xx}>0$ → local min
D>0 & $f_{xx}<0$ → local max
D<0 → saddle point
D=0 → inconclusive
λ>0 → function curves down toward saddle
λ<0 → func. curves up away from saddle

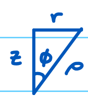
HESSIAN: $H = \begin{bmatrix}f_{xx}&f_{xy}\\f_{yx}&f_{yy}\end{bmatrix}$ $H\vec{v} = \lambda\vec{v}$

for min/max!
LAGRANGE MULT:
1) find CP: $\nabla f = 0$ (within graph)
2) $\nabla f = \lambda\nabla g$ → solve x, y, λ
3) plug all (x,y) into f to find min/max

POLAR:
$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$dA = r\,dr\,d\theta$



CYLINDRICAL:
$dV = r\,dr\,d\theta\,dz$
SPHERICAL:
$dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$



Spherical → cylin.
$z = \rho\cos\phi$
$r = \rho\sin\phi$
$\rho = \sqrt{z^2+r^2}$
$\phi = \tan^{-1}\left(\frac{r}{z}\right)$

rect → sph:
$\rho = \sqrt{x^2+y^2+z^2}$
$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

AVG VAL:
1D: $\frac{1}{b-a}\int_a^b f(x)\,dx$
2D: $\frac{1}{\text{Area}}\iint_R f(x,y)\,dA$
3D: $\frac{1}{\text{Vol.}}\iiint_V f(x,y,z)\,dV$

COM: (same for $\bar{y}$)
$\bar{x} = \frac{1}{M}\iint_R x\rho(x,y)\,dA$
$M = \iint_R \rho(x,y)\,dA$

CHANGE OF VARS:
$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = |J^{-1}| = \frac{1}{|J|}$ det.
1) express x & y in u & v
2) compute Jacobian → $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\\frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}$
3) $\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$
& calculate new bounds
4) evaluate!

DIVERGENCE: (field spread) $\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

CURL: (field rot) $\nabla\times\vec{F} = \det\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\P&Q&R\end{vmatrix}$

LINE INT: $\int_C\vec{F}\cdot d\vec{r} = \int_a^b\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$

GRAD: $\vec{F} = \nabla f = \left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$

CONSERVATIVE:
$\vec{F} = \nabla f$ for some scalar pot. func. f
· line int = path ind.
$\oint_C\vec{F}\cdot d\vec{r} = 0$ around any closed path
· for 2D: if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

conservative ⟷ **path-independent** ⟷ **circulation-free** ⟷ **no curl**
$\int_C \overbrace{\vec{F}}^{\nabla f}\cdot d\vec{r} = f(\vec{r}(B)) - f(\vec{r}(A))$
line int b/t any two pts A & B is same
$\oint_C\vec{F}\cdot d\vec{r} = 0$
$\nabla\times\vec{F} = 0$
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$
closed path:
$\int_C\vec{F}\cdot d\vec{r} = 0$ grad field if curl $\vec{F} = \left\langle\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle = 0 \to Q_z = R_y$, $P_z = R_x$, $P_y = Q_x$

to find f:
$\nabla f = \vec{F}$, or $\frac{\partial f}{\partial x} = P(x,y)$, $\frac{\partial f}{\partial y} = Q(x,y)$
$f(x,y) = \int P(x,y)\,dx + C(y)$
$f(x,y) = \int P(x,y)\,dy + C(x)$
f = terms (union)

line int → double
GREENS THM: $\oint(P\,dx + Q\,dy) = \iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$, $\vec{F} = \langle P, Q\rangle$
C: + closed curve
$\vec{F} = (P(x,y), Q(x,y))$
R: enclosed region
BREAKS DOWN:
· C not closed
· $\vec{F}$ not defined/not smooth
orient: ccw

FLUX: $\oint\vec{F}\cdot\hat{n}\,dS$
$\hat{n}$ = outward unit norm. vector of T
dS = arc len (2D), surface area (3D)

GREENS: $\oint_C\vec{F}\cdot\vec{n}\,ds = \iint_R\nabla\cdot\vec{F}\,dA = \iint_R\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA$

SURFACE: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

SURFACE AREA: $A = \iint_D\left\|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\|du\,dv$

STOKES THM: $\iint_S(\nabla\times\vec{F})\cdot d\vec{S} = \oint\vec{F}\cdot d\vec{r}$
C: ∂S
$\vec{F}$ = vect. field
BREAKS:
· surface/field not smooth
· curve C not oriented consistent
orient: use RHR!

flux int → trip.
DIV THM: $\int_{\partial T}\vec{F}\cdot d\vec{S} = \int_T\overbrace{(\nabla\cdot\vec{F})}^{\text{div}}dV$
$\vec{F}$ = continuous, diff vect. field
S = ∂E orient: norm vects. out
BREAKS:
· surface not closed
· $\vec{F}$ has singularities in vol.

LINE INT: $\int_C\vec{\nabla}f\cdot d\vec{r} = f(B) - f(A)$ or $\int_C\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f(B) - f(A)$

region
(vector form) (classical differential form)
GREEN'S THM: (2D) $\int_R \text{curl}\vec{F}\,dA = \oint_{\partial R}\vec{F}\cdot d\vec{r}$ or $\int_R\underbrace{(Q_x - P_y)\,dx\,dy}_{2\text{ form}} = \oint_{\partial R}\underbrace{\overbrace{P\,dx + Q\,dy}^{w}}_{1\text{ form}}$

surface S, C = ∂S
STOKE'S THM: (3D) $\int_S(\text{curl}\vec{F})\cdot\hat{n}\,d\vec{S} = \oint_{\partial S}\vec{F}\cdot d\vec{s}$ or $\int_S\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$
$= \oint P\,dx + Q\,dy + R\,dz$

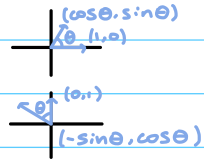
oriented outwards
DIVERGENCE THM: $\int_T(\text{div}\vec{F})\,dV = \oint_{\partial T}\vec{F}\cdot d\vec{s}$ or $\int_T\underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz}_{3\text{ form}} = \oint_T\underbrace{P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy}_{2\text{ form}}\ w$
space region T, ∂T = S

Standard basis vectors: $\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in 2D

Ex: <u>rotation by θ:</u>

$R(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$R(\theta)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$



$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Ex: <u>reflect across x-axis:</u>

$R_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ unchanged

$R_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

dependence:

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \vec{0}$ if indep.

$c_1 + 2c_2 = 0 \rightarrow c_1 = -2c_2$

$2c_1 + 4c_2 = 0 \rightarrow 2(-2c_2) + 4c_2 = 0 \rightarrow \underline{0=0}$

<u>lin. dep!</u>

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] R2-2R_1 \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

eigen:

$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

$\det\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$12 - 7\lambda - 2 - \lambda^2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$(\lambda-5)(\lambda-2) = 0$

$\lambda = 2$:

$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{matrix} 2x+2y=0 \\ x+y=0 \end{matrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 5$:

$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{matrix} -x+2y=0 \\ x-2y=0 \end{matrix}$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

roots:

Ex: want to find r, θ s.t. $(re^{i\theta})^n = \_e^{i\_}$

$r^n e^{i\theta n} \rightarrow r = \sqrt[n]{r}$

$\theta = \dfrac{\theta + 2\pi k}{n}$

Ex: $(1+i)^5 = \sqrt{2}^5 e^{5i\pi/4} = 4\sqrt{2}e^{5\pi/4}$

$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -4-4i$

volume - shadows

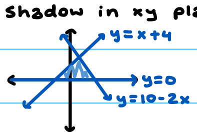
$z=-6$, $y=0$, $y-x=4$, $2x+y+z=4$

$z=-6 \rightarrow z = 4-y-2x$

set eq:

$-6 = 4-y-2x \rightarrow y = 10-2x$

shadow in xy plane:



bounds:

$y: 0 \rightarrow 6$

$x: y-4 \rightarrow \frac{1}{2}(10-y)$

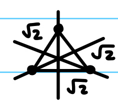
(horiz. slices)

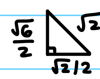
$$V = \int_0^6 \int_{y-4}^{\frac{1}{2}(10-y)} \int_{-6}^{4-2x-y} dz\,dx\,dy$$

Stokes - 3D

triangle path (1,0,0), (0,1,0), (0,0,1) & back.

$\vec{F} = \langle y, -z, x \rangle$, calculate circulation of $\vec{F}$ around C





area $= \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$

$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 1$

$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = -1$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1$

curl: $\langle 1, -1, -1 \rangle$

$\hat{n}: \frac{1}{\sqrt{3}}\langle 1, -1, -1 \rangle$

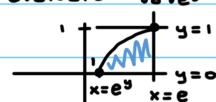
<u>Stokes:</u>

curl $\cdot\, d\vec{S} = \langle 1, -1, -1 \rangle \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2} \cdot \hat{n} = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} = \boxed{\frac{1}{2}}$

order of integration

evaluate $\int_0^1 \int_{e^y}^{e} \frac{x^2}{\ln x}\,dx\,dy \rightarrow \int_1^e \int_0^{\ln x} \frac{x^2}{\ln x}\,dy\,dx$



<u>new bounds:</u>

$y: 0 \rightarrow \ln x$

$x: 1 \rightarrow e$

$= \cdots$

tan plane surface $xz^3 - y + z = 3$

are there pts where tan plane is $\perp$ to xz-plane?

* planes are // if normal vects are //!

xz-plane: normal $\hat{n} = \langle 0, 1, 0 \rangle$

tan plane:

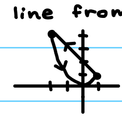
$(\nabla f)(P)$ is $\perp$ to tan plane @ P

$\vec{\nabla} f = \langle z^3, -1, 3xz^2+1 \rangle = \langle 0, c, 0 \rangle$

$\left.\begin{matrix} z^3 = 0 \\ -1 = c \\ 3xz^2+1 = 0 \end{matrix}\right] \times$ no such tan plane!

Green's Thm: $\vec{F}$ along C

start $(-2,4)$, parabola $y=x^2$ to pt $(1,1)$ then straight line from $(1,1)$ back



$C = \partial R$

orient: ccw ✓

Greens (2D):

$\vec{F}(x,y) = \langle x, x^2+y \rangle$

$Q_x - P_y = 2x$

$$\int_{-2}^{1} \int_{x^2}^{2-x} 2x\,dy\,dx = \cdots = -4.5$$

flux - direct calc

S_1: disk $x^2+y^2 \leq 25$ @ height $z=5$

calculate flux up through S_1

$\vec{F} = \langle 2x+3, 3y+2, 4z+1 \rangle$

$\vec{F}(x,y,5) = \langle 2x+3, 3y+2, 21 \rangle$

$\hat{n} = \langle 0,0,1 \rangle$ "up through"

$\vec{F} \cdot \hat{n} = 21$

$\int_S \vec{F} \cdot \hat{n}\, dS = 21 \cdot (\text{area of } S)$

$= 21 \cdot \pi(5^2) = \boxed{525\pi}$