CATTU
1806

# EXAM 1

$A = CR$:

Ex: $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$

rank(A) = 2

1st 2 cols = ind

$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$col_3(A) = 1\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}$

$col_4(A) = 2\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\2\end{pmatrix}$

$R = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 2 \end{pmatrix}$

$CR = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$

can also solve:

$col_4$: $x = 2$, $2y = 4$...

**SPAN:** set of all possible vectors formed from linear combos of those vectors → 1D, 2D plane, $\mathbb{R}^3$, etc.

**LEN/DOT. PROD:** $\|\vec{v}\| = \sqrt{x^2+y^2}$, $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$, $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$

**TRIANGLE INEQ:** $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

$\|\vec{u}+\vec{v}\| = (\vec{u}+\vec{v})^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$
$= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$
$= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$
$= (\|\vec{u}\| + \|\vec{v}\|)^2$

**RANK(A):** # linearly independent cols/rows

**ALGEBRA RULES:**
- $A(cB) = c(AB)$
- $A(B+C) = AB + AC$ (distributive)
- $A(BC) = (AB)C$ (associative)
- commutatively doesn't always hold, even for square matrices

UPPER Δ: $\begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}$

LOWER Δ: $\begin{bmatrix} \cdot & 0 & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}$

**LINEAR IND/DEP:** dep. if one vect. can be written in terms of others.
indep. if $a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = \vec{0}$ w/ a's = 0

**SQUARE LINEAR SYS:** rank(A) = full n, ∃ unique soln. $\vec{x}$ to $A\vec{x} = \vec{b}$
if rank(A) < n, either: NONE or ∞ solns.

**FACTOR A = CR**
C = linearly ind. cols
R = combos/I of cols

⊥: $\vec{u}\cdot\vec{v} = 0$
//: $\vec{u}\cdot\vec{v} = 1$

**TRANSPOSE:** reflect across diag.
$(A^T)_{ij} = A_{ji}$  Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
symmetric: $A^T = A$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
skew-symmetric: $A^T = -A$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

**PERMUTATION PROPERTIES:**
- rank(A) = n (invertible)
- $P^T$ also perm. matrix
- rows of P ⊥ to each other
- $P_1P_2$ also a permutation
- $P^{-1} = P^T$

**INVERSE:** $A^{-1}$ s.t. $AA^{-1} = I = A^{-1}A$, for $A \in \mathbb{R}^{n\times n}$
*only square matrices - these all true:
⇔ rank(A) = n
⇔ det(A) ≠ 0
⇔ $A \in \mathbb{R}^n$
⇔ $(A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0})$
⇔ exists B s.t. AB = I
⇔ exists C s.t. CA = I
$[A \mid I] \to [I \mid A^{-1}]$
*use row elimination

**MORE PROPS:**
- if AB = CA = I, B = C
- if AB has inv, so do A & B, and $(AB)^{-1} = B^{-1}A^{-1}$ where A & B are square
- if A & B are square, & AB have full rank n, A & B have rank n

**PA = LU FACTORIZATION**

Ex: $A = \begin{pmatrix} 1 & 2 & a \\ 2 & 4 & b \\ 3 & 7 & c \end{pmatrix}$

swap R2↔R3: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & a \\ 3 & 7 & c \\ 2 & 4 & b \end{pmatrix}$

$R_3 = R_3 - 2R_1$: $E_{31}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & a \\ 3 & 7 & c \\ 0 & 0 & b-2a \end{pmatrix}$

$R_2 = R_2 - 3R_1$: $E_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & c-3a \\ 0 & 0 & b-2a \end{pmatrix}$

$L = E_{32}^{-1}E_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & c-3a \\ 0 & 0 & b-2a \end{pmatrix}$

$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & a \\ 0 & 1 & c-3a \\ 0 & 0 & b-2a \end{pmatrix} = \begin{pmatrix} 1 & 2 & a \\ 3 & 7 & c \\ 2 & 4 & b \end{pmatrix}$

# EXAM 2

$\vec{w}_1 \cdot \vec{w}_1 = \vec{w}_1^T \vec{w}_1$

$B^TB = I$ means: all cols in B are ⊥

rowsp. $C(A) \perp N(A^T)$

$C(A^T) \perp N(A)$

$P = P^T$ always

**VECTOR SPACE:**
1) set of matrices is closed under addition/subtraction
2) contains 0

closed = rank is preserved

$N(A)$ = set of all vectors $\vec{x}$ satisfying $A\vec{x} = 0$
→ ⊥ to rows of A

$N(A^T)$ = vectors all ⊥ to columns of A

**4 FUNDAMENTAL SUBSPACES:** for $A^{m\times n}$, rank = r
$C(A)$: $\mathbb{R}^m$, dim = r  $N(A^T)$: $\mathbb{R}^m$, dim = m − r ← left nullsp.
$C(A^T)$: $\mathbb{R}^n$, dim = r  $N(A)$: $\mathbb{R}^n$, dim = n − r

$\text{proj}_{\vec{a}}\vec{b} = \overset{p\vec{b}}{\vec{p}} = A\hat{x} \longrightarrow \hat{x} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$

**LEAST SQUARES:** the mean

projection $P = A(A^TA)^{-1}A^T$, $A^TA\hat{x} = A^T\vec{b}$, $\hat{x} = (A^TA)^{-1}A^T\vec{b}$

$P\vec{x} = \vec{x}$ for $\vec{x} \in C(A)$, $P\vec{x} = 0$ for $x \in C(A)^\perp = N(A^T)$

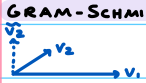
orthonormal $Q \in \mathbb{R}^{n\times k}$ → $\hat{x} = Q^T\vec{b}$, $P = QQ^T$

**ERROR:** $e_i = b_i - p_i$

$Q^TQ = I$

**ORTHONORMAL BASIS:** set of vectors $\vec{v}_i$ that are:
- orthogonal
- unit vectors

**GRAM-SCHMIDT:** give $v_1, v_2, v_3$ a basis
→ outputs $\vec{v}_1, \vec{v}_2, \vec{v}_3$ orthonormal
$v_1$: stays same
$\vec{v}_2$: $v_2 - \text{proj}_{v_1}v_2$
$\vec{v}_3$: $v_3 - \text{proj}_{v_1}v_3 - \text{proj}_{\vec{v}_2}v_3$



**PARTICULAR SOLN:** set free vars = 0

**SPECIAL SOLN:** set one free var = 1 at a time

**COMPLETE SOLN:** $x_c = \begin{bmatrix}\text{part.}\\\text{soln}\end{bmatrix} + c_1\begin{bmatrix}\text{special}\\\text{soln 1}\end{bmatrix} + c_2\begin{bmatrix}\text{special}\\\text{soln 2}\end{bmatrix}$

**INVERSE:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: orthogonal vectors (2, 2, −1) & (−1, 2, 2). Find orthonormal vectors & put into col Q.

orthog: −2 + 4 − 2 = 0 ✓

$v_1 = \frac{1}{3}\begin{bmatrix}2\\2\\-1\end{bmatrix}$, $v_2 = \frac{1}{3}\begin{bmatrix}-1\\2\\2\end{bmatrix}$ → $Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$

$Q^TQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, $QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix} = P$

Ex: for which $(b_1, b_2, b_3)$ is system solveable?

$\begin{pmatrix} 1 & 4 \\ 2 & 9 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ when $b_1 + b_3 = 0$ → $\begin{pmatrix} p \\ q \\ -p \end{pmatrix}$  $p, q \in \mathbb{R}$

$\left(\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & 4 & b_3 \end{array}\right) \xrightarrow{b_2+2b_3} \left(\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2+2b_3 \\ -1 & 4 & b_3 \end{array}\right) \xrightarrow{b_3+b_1} \left(\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2+2b_3 \\ 0 & 0 & b_3+b_1 \end{array}\right)$

Ex: find basis of 4 subspaces for $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix}$

dim = 2
$C(A) = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}\right\}$

dim = 2
$C(A^T) = \text{span}\left\{\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}\right\}$  rank = 2

RREF A: $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ free var: $x_3$ → $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$x_1 + 2x_2 + 4 = 0$
$x_2 = 0$ → $N(A)$ → span$\left\{\begin{pmatrix}0\\-4\\1\end{pmatrix}\right\}$
dim = 3 − 2 = 1

RREF $A^T$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  $N(A^T) \to \vec{0}$
dim = 2 − 2 = 0

Ex: give particular/complete soln to

Ex: gram-schmidt

Ex: $S = \text{span}\{[1\ 2\ 2\ 3], [1\ 3\ 3\ 2]\}$. Find 2 vectors that span $S^\perp$ → set of all vectors ⊥ to every S vector

$S^\perp = N(A)$ where A's rows are spanning vectors of S. ∴ N(A) gives $S^\perp$

$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}\vec{y} = 0 \to \begin{matrix} y_1 + 2y_2 + 2y_3 + 3y_4 = 0 \\ y_1 + 3y_2 + 3y_3 + 2y_4 = 0 \end{matrix}$

$y_2 = \overset{-s}{-y_3} + \overset{+t}{y_4}$  free vars: $\overset{s}{y_3}, \overset{t}{y_4}$

$y_1 = -5y_4$ → $y = s(0\ -1\ 1\ 0) + t(-5\ 1\ 0\ 1)$
$s = y_3 = 1, t = 0$  $t = 1, s = 0$

span = $\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\1\\0\\1\end{bmatrix}\right\}$

Ex: b = (0 8 8 20), t = (0 1 3 4) solve eqns & errors E

$p = A\hat{x}$
$p(t) = C + Dt$ → $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$  $A^TA\hat{x} = A^Tb$

$A^TA = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$  $A^Tb = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ → $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\hat{x} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ → $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$p(t) = 1 + 4t$ → $p(0) = 1$, $p(1) = 5$, $p(3) = 13$, $p(4) = 17$

errors $e_i = b_i - p_i$: $e_1 = -1$, $e_2 = 3$, $e_3 = -5$, $e_4 = 3$

Ex: orthog vectors from: (1, −1, 0, 0), (0, 1, −1, 0), (0, 0, 1, −1)

$\vec{A} = \frac{1}{\sqrt{2}}(1, -1, 0, 0)$

$\vec{B} = (0, 1, -1, 0) - \text{proj}_A B = (0, 1, -1, 0) + \frac{1}{2}(1, -1, 0, 0) \to \vec{B} = (1/2, 1/2, -1, 0)$

$\vec{C} = (0, 0, 1, -1) - \overset{0}{\text{proj}_A C} - \text{proj}_B C = (0, 0, 1, -1) + \frac{2}{3}(1/2, 1/2, -1, 0) \to \vec{C} = (1/3, 1/3, 1/3, -1)$

**DETERMINANT:**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-fh) - b(di-gf) + c(dh-eg)$$

AREA in $\mathbb{R}^2 = |\det(\vec{v}_1\ \vec{v}_2)|$

VOL in $\mathbb{R}^3 = |\det(\vec{v}_1\ \vec{v}_2\ \vec{v}_3)|$

**TRACE:** sum of $\lambda$'s

$tr(AB) = tr(BA)$

$tr(A) = tr(X\Lambda X^{-1}) = tr(\Lambda) = \lambda_1 + \lambda_2 + \dots$

**COFACTOR:** $C_{ij} = (-1)^{i+j}|M_{ij}|$

Example: Find the cofactor matrix of **A** given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

Solution: First find the cofactor of each element.

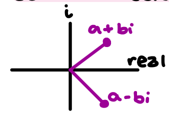
$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \quad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \quad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$

$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$

The cofactor matrix is thus $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$.

**COMPLEX CONJUGATION:** for $z = a+bi$ & $\bar{z} = a-bi$,




norm: $|a+bi| = \sqrt{a^2+b^2}$

$z\bar{z} = (a+bi)(a-bi) = a^2+b^2 = |z|^2$

$\overline{z+w} = \bar{z} + \bar{w}$, $\quad \overline{zw} = \bar{z}\bar{w}$

**EULER:** for $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow |e^{i\theta}| = 1$

**EIGENVALUE $\lambda$:**

- square matrix $A$ s.t. $A\vec{v} = \lambda\vec{v}$
- $I \to \lambda = 1$
- some $A$ have no real $\lambda$
- if $N(A) \neq \{0\}$, $\lambda = 0$ is eval of $A$

**PRESERVED:** matrix $A$ preserves subspace $V$ if $\vec{v} \in V$ & $A\vec{v} \in V$

**SYMMETRIC MATRIX:** $A = A^T$

- all $\lambda$ are real
- always $n$ orthonormal e.vects
- can always be diagonalized by $\perp$ matrix

**SPECTRAL THM:** $S = S^T \to \exists$ orthonormal

$Q = (\vec{v}_1, \dots, \vec{v}_n)$ cols = normalized e. vects

$SQ = Q\Lambda$, $Q^{-1} = Q^T \Rightarrow S = Q\Lambda Q^{-1}$

**DIAGONALIZATION:**

$AX = X\Lambda \to A = X\Lambda X^{-1}$

$\Lambda$: $\lambda$'s on diagonals

$X$: eigenvects on cols

$X$ invertible $\to A = X\Lambda X^{-1}$

$A^m = X\Lambda^m X^{-1}$

**HERMATIAN MATRIX:**

$A \in \mathbb{C}^{n\times n}$ is hermatian if $A^* = A$

(complex analogue of real symm)

- e. vals are real
- e. vecs for distinct evals are $\perp$

inner prod: $\vec{v}\cdot\vec{w} = \bar{\vec{w}}^T\vec{v} = \bar{w}_1 v_1 + \bar{w}_2 v_2 + \dots$

$\to$ makes lengths real and positive

Ex: $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix} \to \underset{\text{conjugate}}{\begin{pmatrix} 2 & 3+3i \\ 3-3i & 5 \end{pmatrix}} \to \underset{\text{transpose}}{\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}}$ =

**UNITARY MATRIX:**

if $U^*U = I_n \Leftrightarrow U^* = U^{-1}$

- preserve len/angle

$U^*$ = conjugate transpose

**QUAD FORM:** $Q(x) = x^TAx$

Ex: $A = \begin{array}{c|cc} & x_1 & x_2 \\ x_1 & 3 & 2 \\ x_2 & 2 & 1 \end{array}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q(x) = 3x_1^2 + 2(2)x_1x_2 + x_2^2$

**MATRIX POWERS:** $e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots$, $t$ scalar

$u(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2 + \dots$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to e^A = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$e^{tA} = I + tA = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$e^{tA}e^{-tA} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = I$

$z = e^{2\pi ki/n}$ for $n$ = # of roots

**PROPERTIES:**

1) $\frac{d}{dt}e^{tA} = Ae^{tA}$
2) $(e^{tA})^{-1} = e^{-tA}$
3) if $A^T = -A$ (skew sym) $\to e^{tA}$ is orthog.
4) $A\vec{x} = \lambda\vec{x} \to e^{tA}\vec{x} = e^{t\lambda}\vec{x}$

**DIFFERENTIAL EQNS:**

$f''(t) = af(t) + bf'(t)$ $\quad a, b \in \mathbb{R}$

$\vec{u}(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} \quad \vec{u}'(t) = \begin{pmatrix} f'(t) \\ f''(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}}_{\vec{u}(t)}$

*bottom row must = og equat.

$\vec{u}' = A\vec{u} \to \vec{u}(t) = e^{tA}\vec{u}(0)$

$\vec{u}'(t) = Ae^{tA}\vec{u}(0)$

$f'''(t) = af(t) + bf'(t) + cf''(t)$, $a, b, c \in \mathbb{R}$

$\vec{u}(t) = \begin{pmatrix} f(t) \\ f'(t) \\ f''(t) \end{pmatrix} \quad \vec{u}'(t) = \begin{pmatrix} f'(t) \\ f''(t) \\ f'''(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}\begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} \quad \vec{u}(t) = e^{tA}\vec{u}(0)$

**SINGLE VALUE DECOMPOSITION:** $A = U\Sigma V^T$ ($m\times m$, $m\times n$, $n\times n$), $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$

$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \end{pmatrix} \quad \sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2} \quad U^TU = I$, $V^TV = I$

$V^T$: cols = trans. normalized e. vects. of $A^TA$ $\quad (A^TA - \lambda_1 I)v_1 = 0$

$U$: cols = normalized e. vects. of $AA^T$ $\quad u_i = \frac{1}{\sigma_i}Av_i \quad v_i = \frac{1}{\sigma_i}A^Tu_i$

$A = \sigma_1u_1v_1^T + \sigma_2u_2v_2^T + \dots + \sigma_ru_rv_r^T$, $r$ = rank

**RANK K-APPROXIMATION OF A:** $A_k = \sigma_1u_1v_1^T + \dots + \sigma_ku_kv_k^T$

for any rank $k$ matrix $B$, $\|A - A_k\| \leq \|A - B\|$

**ECKART-YOUNG:** if $\text{rank}(B) = k$, $\|A - B\|_F \geq A - (\sigma_1\vec{u}_1\vec{v}_1^T + \dots + \sigma_k\vec{v}_k\vec{u}_k^T)\|_F$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \to$ ($\lambda$ & e vects) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \to (2-\lambda)^2 - 1 = 3 - 4\lambda + \lambda^2 = 0 \to \lambda = 3, 1$

$\lambda = 1$: $A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$A^{100}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Ex: $A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$ $\leftarrow$ Hermatian

$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = 10 - 7\lambda + \lambda^2 - (9 - -9) = \lambda^2 - 7\lambda - 8 = 0 \to \lambda = -1, 8$

$\lambda = -1$: $A + I = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix} \to \vec{v}_1 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$

$\lambda = 8$: $A - 8I = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix} \to \vec{v}_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

2 lin. ind. e. vects

Ex: $n = 1$ $\quad u = e^{i\theta}$ $\quad$ unitary matrix

$\bar{u} = e^{-i\theta}$

$u\bar{u} = e^{i\theta}(e^{-i\theta}) = 1 = I$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad$ matrix powers

$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$, $A^3 = A^2A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A$, $A^4 = A^3A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, $A^5 = A \dots$

*cycle of 4 repeats

$e^{At} = I + At + \frac{t^2}{2}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!}(I) + \dots$

$= I\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots\right) + A\left(t - \frac{t^3}{3!} + \dots\right)$

$e^{tA} = I\cos t + A\sin t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

Ex: $f(t)$ solves $\begin{cases} f''(t) = -f(t) \\ f(0), f'(0) \text{ given} \end{cases}$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad$ diff. eqns.

$\vec{u} = \begin{pmatrix} f \\ f' \end{pmatrix} \quad \vec{u}' = \begin{pmatrix} f' \\ f'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} f \\ f' \end{pmatrix} \to$ soln: $u(t) = e^{tA}\vec{u}(0)$

$u(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$

Ex: solve $\frac{du}{dt} = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}u$, $u(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

... find $\lambda = 0$, $\lambda_2 = -5$

$\lambda = 0$: $\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to y = \frac{2}{3}x$, $v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda = -5$: $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to y = -x$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2$

$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \begin{matrix} c_1 = 1 \\ c_2 = 1 \end{matrix} \to u(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Ex: $A = \begin{bmatrix} 1+i & 2 \\ 3i & -4i \end{bmatrix}$

$A^*$ = conjugate trans = $\begin{bmatrix} 1-i & -3i \\ 2 & 4i \end{bmatrix}$

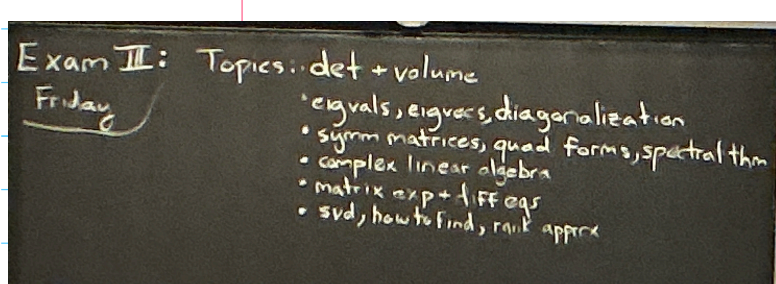
not Hermatian b/c $A \neq A^*$

Ex: $v = [1+i, 2]$, $w = [3, i]$

$\bar{w} = [3, -i]$

$\langle v, w\rangle = (1+i)(3) + (2)(-i) = 3 + i$ $\leftarrow$ Hermatian inner prod

$\|v\|^2 = \langle v, v\rangle = \bar{v}^Tv = (1-i)(1+i) + 2\cdot2 = (1+1) + 4 = 6$ $\quad \|v\| = \sqrt{6}$



# 18.06 Midterm 3: crash course

## Determinant + Vol

Ex: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \det = ad - bc$

Ex: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \to \det = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

**Cramers:** $A\vec{x} = \vec{b}$ $\quad x_i = \frac{\det A_i}{\det A}$

where $A_i$ has b as ith col

## Eigenvalues, Eigenvectors

$A\vec{v} = \lambda\vec{v}$ $\quad |A - \lambda I|$ to find $\lambda$s

$(A - \lambda I)\vec{v} = \vec{0}$ to find $\vec{v}$s

$\det(A) =$ product of $\lambda$

$tr(A) =$ sum of $\lambda$

## Diagonalization

$e^{tA} = Xe^{t\Lambda}X^{-1}$

$A = V \Lambda V^{-1}$ $\quad$ if symm: $A^T = A^{-1}$

$V$: cols = e. vects $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$\Lambda$: diagonals = $\lambda$ $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

- not orthogonal so don't need norm.

## Matrix Exponential:

$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \ldots$

$t$ = scalar $\quad A$: must get powers

## SVD:

$A = U\Sigma V^T$ $\quad U^TU = I, V^TV = I$ (orthog)

- $U$: cols = normalized e. vects of $AA^T$ ($n \times n$)
- $\Sigma$: diag = $\sigma = \sqrt{\lambda\text{'s}}$ starting from greatest $\lambda$ ($n \times m$ careful of dims!)
- $V^T$: rows = normalized e. vects of $A^TA$ ($m \times m$)

$v_i = \frac{1}{\sigma_i}A^Tu_i$ $\quad u_i = \frac{1}{\sigma_i}Av_i$

starting from max $\sigma$

## Rank k Approx:

$B = \sigma_1 u_1 v_1 + \ldots + \sigma_r u_r v_r$

## Eckart - Young Theorem:

if $B$ has rank$(B) = k$, then $\|A - B\|_F \geq \|A - (\sigma_1 \vec{u}_1 \vec{v}_1^T + \ldots + \sigma_k \vec{u}_k \vec{v}_k^T)\|_F$

* single best choice is the first k terms?

## Symmetric Matrix: $A = A^T$

- all real $\lambda$'s
- e. vects are orthog.
- use Spectral Thm

## Spectral Thm: $A = Q\Lambda Q^T$

- $Q^T = Q^{-1}$ b/c symmetric!
- $Q$: cols = normalized e. vects
- $\Lambda$: diag = $\lambda$'s

## Quad Form: $Q(t) = x^TAx$

$ax^2 + 2bxy + cy^2$ $\quad \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 2 & -1 & 0 \\ x_2 & -1 & 2 & -1 \\ x_3 & 0 & -1 & 2 \end{array} = S \to x^TSx = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

## Definity:

positive definite: $x^TAx > 0$, $\lambda$'s $> 0$, $\quad$ for + definite: all dets of squares $> 0$

positive semi-def: $x^TAx \geq 0$, $\lambda$'s $\geq 0$

## Differential Equation:

$A\vec{x} = \lambda\vec{x} \Rightarrow e^{tA}\vec{x} = e^{t\lambda}\vec{x}$

$x(t) = c_1e^{\lambda_1 t}v_1 + \ldots + c_ne^{\lambda_n t}v_n$

$x(t) = e^{tA}x(0)$

either

## Complex #'s: Hermatian

inner prod: $\vec{v} \cdot \vec{w} = \overline{\vec{w}}^T \vec{v} = \overline{w}_1 \cdot v_1 + \overline{w}_2 \cdot v_2 + \ldots$

- real $\lambda$'s
- e. vects are orthogonal
- "complex version of symm. matrix"
- $A = A^*$ where $A^*$ = conjugate transpose of $A$ $\bar{A}^T$

## Unitary: $U^* = U^{-1} \iff U^*U = I_n$

- preserves len/angle
- $U^*$ = conjugate transpose
- $\lambda$'s have magnitude 1
- unit eigenvectors

CONTENT BEFORE FINAL:

**LINEAR TRANSFORMATION:** is map $T: V \to W$ with

a) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

b) $T(\alpha\vec{v}) = \alpha T(\vec{v})$ & $T(\vec{0}) = \vec{0}$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ $T(x) = Ax \to$ 1st comp: $2x_1 + x_2$, 2nd comp: $3x_2$

$x = (1,4)$ $T(x) = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

**STOCHASTIC:** $\vec{p}$ is stochastic if

1) all entries $\geq 0$

2) $\vec{p}^T\vec{1} = 1 \to \vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ column sums to 1. encodes probabilities

**MARKOV:** $A \in \mathbb{R}^{n \times n}$ is Markov if every col is Stochastic

$A = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ encodes transition probs.

col$_j A$ = probability given in state $j$. column sums to 1

$\vec{P}_j$ = probability of being in state $j$

$\lambda$ contains 1

$A^k\vec{p} \xrightarrow[k\to\infty]{}$ long term prob. for being in each state

$A^k\vec{p} = c_1 A^k \vec{v}_1 + \ldots$

**CHANGE OF BASIS:**

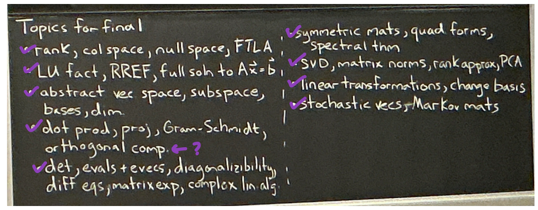
new input $V = (\vec{v}_1 \cdots \vec{v}_n)$

new output $W = (\vec{w}_1 \cdots \vec{w}_m)$

new matrix is $W^{-1}AV$

to do:

- ☑ study past content (notes)
- ☑ past exams
- ☑ past practice exams (earlier tests)
- ☑ past rec
- ☑ practice exams (final)
- ☑ read through notes
- ☑ create final c.s.




FINAL CHEAT SHEET:

**RANK:** # of independent cols = # independent rows
# of pivots in RREF

**COL SPACE C(A):** span of matrix cols (lin. ind. cols)

**NULL SPACE N(A):** vectors where $A\vec{x}=\vec{0}$

**FTLA:** $\dim(C(A)) = r$, lives in $\mathbb{R}^m$ $\quad A = m\overset{n}{[\quad]}$

$\dim(N(A)) = n-r$, lives in $\mathbb{R}^n$

$\dim(C(A^T)) = r$, lives in $\mathbb{R}^n$ $\quad A^T = n\overset{m}{[\quad]}$

$\dim(N(A^T)) = m-r$, lives in $\mathbb{R}^m$

row space $C(A^T) \perp N(A)$,
$C(A) \perp$ left nullsp $N(A^T)$

**BASIS:** set of vectors that
1) spans the space
2) is linearly independent ($c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$)
dim = # basis vectors

**DOT PRODUCT:** $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$ scalar!
- $u \cdot v = \|u\|\,\|v\|\cos\theta$ → tells angle b/t vectors
- $u \cdot v = 0$ → u & v ⊥

**PROJECTION:** $\text{proj}_a b = \frac{a\cdot b}{a\cdot a}\vec{a}$ onto vector a
$P = A(A^TA)^{-1}A^T$ onto subspace spanned by A

**DET:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → $\det A = ad - bc$
- area/vol scaling factor
- if $\det A = 0$ → A is not invertible

**DIFFERENTIAL EQUATIONS:**
$u'(t) = Au(t)$
$u(t) = e^{\lambda t}v$ → if A diagonalizable: $u(t) = \sum c_i e^{\lambda_i t}v_i$
★ $x(t) = c_1e^{\lambda_1 t}v_1 + \dots + c_ne^{\lambda_n t}v_n$
$x(t) = e^{tA}x(0)$ ← if diagonalizable
$\vec{u} = \begin{pmatrix} f \\ f' \end{pmatrix}$ $\quad \vec{u}' = \begin{pmatrix} f' \\ f'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}}_{A}\begin{pmatrix} f \\ f' \end{pmatrix}$ → $u(t) = e^{tA}\vec{u}(0)$

★ **COMPLEX LIN ALG:** $\lambda = ai + b$
$e^{i\theta} = \cos\theta + i\sin\theta$

**HERMATIAN:** $U^* = \bar{U}^T$
- real λ's
- inner prod $\vec{v}\cdot\vec{w} = \bar{\vec{w}}^T\vec{v} = \bar{w}_1v_1 + \dots$
- e. vecs orthog.

**UNITARY:** $U^* = U^{-1} \leftrightarrow U^*U = I_n$
- preserves len/angle
- $U^*$ = conjugate transpose

**SVD:** $\underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$: $A = U\Sigma V^T$, $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$
$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \sigma_2 & \\ & & \ddots \end{pmatrix}$ $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$ decr. order
$V^T$ = cols = transposed norm. e.vects. of $A^TA$
U = cols = norm. e.vects of $AA^T$
$A = \sigma_1u_1v_1^T + \dots + \sigma_ru_rv_r^T$, r = rank
$u_i = \frac{1}{\sigma_i}Av_i$ $\quad v_i = \frac{1}{\sigma_i}A^Tu_i$ $\quad U^TU = I$, $V^TV = I$

**MATRIX NORM:**
- Frobenius $\|A - A_k\|_F = \sqrt{\sigma_{k+1}^2 + \dots + \sigma_r^2}$ of SVD s.v.s

**RANK APPROX:** $A_k = \sigma_1u_1v_1^T + \dots + \sigma_ku_kv_k^T$
for any rank k matrix B, $\|A - A_k\| \le \|A - B\|$

**CHANGE OF BASIS:**
1) express $T(v_1), T(v_2) \dots$ as linear combo of new basis vectors
2) coefficients become cols of new matrix
input basis $v = (\vec{v}_1 \cdots \vec{v}_n)$
output basis $w = (\vec{w}_1 \cdots \vec{w}_n)$ $\quad M = \underset{\text{out}}{W^{-1}}\underset{\text{trans.}}{A}\underset{\text{input}}{V}$

W converts from new → standard
$W^{-1}$ converts from standard → new

- vector written in basis V
- transformation A
- want output written in basis W

Ex: $V = \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \\ 1 & 1 \end{pmatrix}$
basis vect. as cols
$V\begin{pmatrix} a \\ b \end{pmatrix} = x$
converts coords in new basis

**RREF:** gives you rank, pivot/free cols, null space, full soln to $A\vec{x}=\vec{b}$

**LU FACTORIZATION:** $A = LU$
L = lower △ (multiplier), $E^{-1}$
U = upper △, 0's in bottom left
Ex: $U = \begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \end{bmatrix}$

**FULL SOLN to $A\vec{x}=\vec{b}$:** $x = x_p + x_n$
$x_p$ = particular soln (set free vars = 0, solve for $=\vec{b}$)
$x_n$ = special soln (set free var = 1 one @ a time, solve for $=\vec{0}$)
1) row augment [A|b]
2) solve for pivots in terms of free vars
3) seperate $x_p$ & $x_n$

**SPAN:**
- all linear combos
- smallest subspace containing these vectors

**VECTOR SPACE:**
1) set of matrices is closed under add/sub
2) contains 0 $\quad$ rank preserved

**SUBSPACE:** smaller vector space inside bigger one
a set W is a subspace IF:
1) $0 \in W$
2) $u, v \in W$, then $u + v \in W$
3) $u \in W$ and $c \in \mathbb{R}$, then $cu \in W$

**GRAM-SCHMIDT:** given $v_1, v_2 \dots$
$q_1 = \frac{v_1}{\|v_1\|}$ ← orthonormal
$q_2 = \frac{u_2}{\|u_2\|}$ $\quad u_2 = v_2 - (q_1\cdot v_2)q_1$
$q_3 = \frac{u_3}{\|u_3\|}$ $\quad u_3 = v_3 - (q_1\cdot v_3)q_1 - (q_2\cdot v_3)q_2$

**DIAGONALIZABILITY:** $A = S\Lambda S^{-1}$
Λ = diagonals are λ's
S = cols are e.vecs
- diagonalizeable if λ's distinct

**EIGENVALS/EIGENVECTS:** $Av = \lambda v$, $v \neq 0$
$\det(A - \lambda I) = 0$ $\quad A^xv = \lambda^xv$
- if λ < 0 → decay, λ > 0 → growth

★ **MATRIX EXP:** $e^{tA} = I + tA + \frac{(tA)^2}{2!} + \dots$
if A diagonalizable: $e^{tA} = Se^{t\Lambda}S^{-1}$ where $e^{t\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$
S = eigenvectors

**SYMMETRIC MATRICES:** $A^T = A$
- all λ's are real
- e.vects. to diff. e.vecs are orthog.

**QUAD FORM:** $Q(x) = x^TAx$ $\quad$ x = variables (for polynomials)
- A must be symmetric

★ **SPECTRAL THM:** every real, symmetric matrix can be diagonalized by an orthogonal matrix $A = Q\Lambda Q^T$, $Q^{-1} = Q^T$
Λ = diagonals are λ's
$Q = (\vec{v}_1, \dots, \vec{v}_n)$ cols = normalized e. vects.

★ **PCA:** uses SVD to find directions of max variance
- center matrix rows by subtracting row mean
- do SVD $U\Sigma V^T$ → the U part $AA^T$
- find $S = \frac{1}{n-1}AA^T$ (find λ, e.vects) → n = # cols of A
- cols of U = principal directions
 - 1st col = dir. w/ max variance
- project data onto that reduces dim. while preserve most info

**LINEAR TRANSFORMATIONS:** $\mathbb{R}^n \to \mathbb{R}^m$ is a rule that satisfies
1) $T(u+v) = T(u) + T(v)$
2) $T(cu) = cT(u)$ $\quad$ matrix representation: $T(x) = Ax$
3) $T(\vec{0}) = 0$

**STOCHASTIC VECTORS:**
- non-neg $\quad$ Ex: $\begin{pmatrix} 0.2 \\ 0.5 \\ 0.3 \end{pmatrix}$
- sums to 1

**MARKOV MATRICES:** square matrix P where each col is stochastic vector
- model transitions
- has an λ = 1
- steady state: solve $(P - I)u$, u adds to 1 (MUST NORMALIZE) add to 1
 ↳ what does $A^k\vec{p}$ approach as k → ∞
Ex: $\lambda_1 = 1 \to v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \to \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

EXAMPLE PROBLEMS:

**RREF, FTLA** Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}$

RREF: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ 2 pivots, rank = 2

$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$ dim = r = 2 ✓

$N(A)$: $Ax = 0$, RREF: $\begin{cases} x_1 - x_3 = 0 \\ -x_2 - 2x_3 = 0 \end{cases} \rightarrow$ $N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ dim = 3 − 2 = 1 ✓

rowspace = span of nonzero RREF rows = span $\{(1, 0, -1), (0, 1, 2)\}$ dim = r = 2

left nullsp = m − r → 3 − 2 = 1 $A^T y = 0$

**LU-FACT** Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = U$, $E^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = L$

$LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = A$

ORTHOG COMP EX: $W = \text{span}\{(1,1,0)\}$

$W^\perp = \{(x,y,z) : x + y = 0\}$

**FULL SOLN** Ex: solve $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 2 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$ pivots: $C_1$ & $C_3$

equations: $\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 1 \end{cases}$

$x_p$: $x_2 = 0 \rightarrow x_1 = 0, x_3 = 1$ $x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases}$

$x_s$: $x_2 = 1 \rightarrow x_1 = -2, x_3 = 0$ $x_c = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ full soln: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

**SUBSPACE** Ex: $W = \{(x,y,z) : x + y + z = 0\}$

$\left.\begin{array}{l} 0 \in W ✓ \\ \text{add/sub closed} ✓ \\ \text{scaling closed} ✓ \end{array}\right] \therefore$ subspace

EX: $x + y + z = 1$
no zero vect
$\therefore$ not subspace

**GRAM-SCHMIDT** Ex: $v_1 = (1,1)$, $v_2 = (1,0)$

$q_1 = \frac{(1,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{\frac{2}{4}}} u_2 = \sqrt{2} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

$u_2 = (1,0) - \left((1/\sqrt{2}, 1/\sqrt{2}) \cdot (1,0)\right) q_1 = (1,0) - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

**exp.** ☆ Ex: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$\det(A - I\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 1 = 0 \rightarrow 8 - 6\lambda + \lambda^2 + 1 = 0 \rightarrow (\lambda - 3)^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 3$

$A - 3I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = 0 \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → Jordan Form: $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{array}{l} -w_1 + w_2 = 1 \\ w_1 = 0 \rightarrow w_2 = 1 \end{array}$ general evect: $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$S = [v\ w] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$e^{tJ} = e^{3t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

**SVD, Frobenius, rank-approx, PCA** Ex: $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ SVD: $A = U \Sigma V^T$

$AA^T = \begin{bmatrix} 5 & 3 \\ 3 & 11 \end{bmatrix}$

$|AA^T - \lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 11-\lambda \end{vmatrix} = \overset{46}{55 - 16\lambda + \lambda^2 - 9} \rightarrow \lambda = 8 \pm \sqrt{18}$ $\sigma_1 = \sqrt{8 + 3\sqrt{2}}$, $\sigma_2 = \sqrt{8 - 3\sqrt{2}}$

$\|A_F\| = \sqrt{2^2 + 0 + 1^2 + 1^2 + 3^2 + 1^2} = \sqrt{14} = \sqrt{\sigma_1^2 + \sigma_2^2}$

rank 1 approx: $A_1 = \sigma_1 u_1 v_1^T$

Frobenius error = $\|A - A_1\|_F^2 = \sigma^2$

row 1 mean: 1

row 2 mean: 1.67

] center A, do SVD → 1st col of U = direction of max var, 2nd col = next transform Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y\\x-y\end{pmatrix} \to A = \begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}$

change of basis Ex: $T(x,y) = (2x+y,\ x-y)$ & new basis $v_1 = \begin{pmatrix}1\\1\end{pmatrix},\ v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$T(v_1) = \begin{pmatrix}3\\0\end{pmatrix} \to \begin{pmatrix}3\\0\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix}$ $\quad a+b=3,\ a-b=0 \to a=1.5,\ b=1.5$

$T(v_2) = \begin{pmatrix}1\\2\end{pmatrix} \to \begin{pmatrix}1\\2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix}$ $\quad a+b=1,\ a-b=2 \to a=1.5,\ b=-0.5$

$T_B = \begin{bmatrix}1.5 & 1.5\\1.5 & -0.5\end{bmatrix}$

change of basis Ex: $w_1 = (2,0) \quad w_2 = (0,1)$

change of basis matrix $X = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$

transform Ex: linear $T$ transforms $(1,1)$ to $(2,2)$ & $(2,0)$ to $(0,0)$

$T(v) = c_1 T(1,1) + c_2 T(2,0) = c_1(2,2) + c_2(0,0) \to (2c_1,\ 2c_1)$

$v=(2,2)$: $T(2,2) = c_1(1,1) + c_2(2,0) \to 2 = c_1 + 2c_2,\ 2 = c_1 \to T(v) = (4,4)$

$v=(3,1)$: $T(3,1) = c_1(1,1) + c_2(2,0) \to 3 = c_1 + 2c_2,\ 1 = c_1 \to T(v) = (2,2)$

transform Ex: matrix $B$ transforms $\begin{pmatrix}2\\5\end{pmatrix}$ to $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}1\\3\end{pmatrix}$ to $\begin{pmatrix}1\\0\end{pmatrix}$

means $C = B^{-1} = \frac{1}{6-5}\begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$

if random: $CX = Y$ where $C$ = transform, then $C = YX^{-1}$

transform Ex: matrix that transforms $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}1\\4\end{pmatrix}$ & $\begin{pmatrix}1\\5\end{pmatrix}$

input $X = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, output $Y = \begin{pmatrix}1 & 1\\4 & 5\end{pmatrix}$

$CX = Y,\ C = YX^{-1} = \begin{pmatrix}1 & 1\\4 & 5\end{pmatrix} \cdot 1 \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\4 & 5\end{pmatrix}$

$X^{-1} = \frac{1}{1}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

markov Ex: $A = \begin{pmatrix}0.4 & 0.2\\0.6 & 0.8\end{pmatrix}$ starting state $\vec{p} = \begin{pmatrix}1/2\\1/2\end{pmatrix}$

compute $\lambda$'s $\begin{vmatrix}0.4-\lambda & 0.2\\0.6 & 0.8-\lambda\end{vmatrix} = 0.32 - 1.2\lambda + \lambda^2 - 0.12 = 0 \to \lambda^2 - 1.2\lambda + 0.2 = 0 \to (\lambda-1)(\lambda-0.2) = 0 \to \lambda = 1, 0.2$

steady state for $\lambda = 1$: $A - I = \begin{pmatrix}-0.6 & 0.2\\0.6 & -0.2\end{pmatrix} \to v_1 = \begin{pmatrix}1\\3\end{pmatrix} \to$ normalize $= \frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}$

$A^k$ approaches what matrix $A^\infty$? $A^k$ converges to $\frac{1}{4}\begin{pmatrix}1 & 1\\3 & 3\end{pmatrix}$ since all entries are +

PCA Ex: $A_0 = \begin{bmatrix}5 & 4 & 3 & 2 & 1\\-1 & 1 & 0 & 1 & -1\end{bmatrix}$ ← 2 measurements of 5 samples

CENTER: $r_1$ mean $= \frac{15}{5} = 3$, $r_2$ mean $= 0$

$A_0 = \begin{bmatrix}2 & 1 & 0 & -1 & -2\\-1 & 1 & 0 & 1 & -1\end{bmatrix}$

$A_0^T = \begin{bmatrix}2 & -1\\1 & 1\\0 & 0\\-1 & 1\\-2 & -1\end{bmatrix}$ $\quad AA^T = \frac{1}{4}\begin{bmatrix}10 & 0\\0 & 4\end{bmatrix} = \begin{bmatrix}5/2 & 0\\0 & 1\end{bmatrix}$

find $\lambda$'s: $\lambda = 5/2,\ 1$

for largest $\lambda = 5/2$: $\frac{1}{4}AA^T - \frac{5}{2}I = \begin{bmatrix}0 & 0\\0 & 3/2\end{bmatrix} \to v_1 = \begin{bmatrix}1\\0\end{bmatrix}$

line closest to samples $= \begin{bmatrix}1\\0\end{bmatrix}$