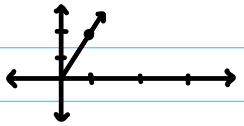
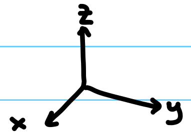
- pset partners
- pset checker
- 48 hr grace - 10% off
- lowest pset drop

9/3/25

1) VECTORS

$\mathbb{R}^2$ ← plane in 2D $\mathbb{R}^3$ 3D ... $\mathbb{R}^n$



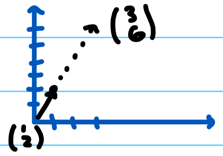


$\begin{pmatrix}1\\2\end{pmatrix}$ ← 1 in x ← 2 in y

WE CAN...

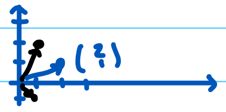
1) **scalar multiply** $a\vec{v}$ (scalar ↳ a, vect. ↙ $\vec{v}$)

ex: $3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix}$ stretch by scalar of 3



2) **add vectors**

ex: $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$



[3 perspectives]

1: ordered list of #'s "algebraic perspective"

2: arrows show dir. + mag. geometric / physics

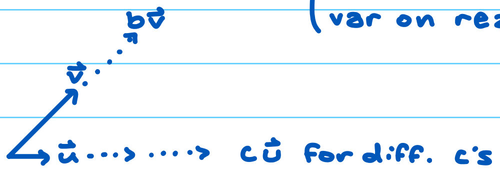
3) "abstract" objects we add/mult (ex: polynom. in one var on real line)

2) LINEAR COMBOS:

all linear combos of 2 vect. $\vec{u}$ & $\vec{v}$:

$\{a\vec{u} + b\vec{v} \mid a, b \in \mathbb{R}\} = \text{span}\{\vec{u}, \vec{v}\}$

all real # ↑



typical q's:

1: what is the span?

2: given $\vec{w}$ in span, how to choose a, b to get it?

3) SPAN / ELIMINATION

Ex: $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\right\} = \mathbb{R}^2$ how to get a, b for vector $\begin{pmatrix}c\\d\end{pmatrix}$?

solve $a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$ (find a, b)

$\begin{pmatrix}a+2b\\a+3b\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$

$$-\begin{cases}a+2b=c\\a+3b=d\end{cases}$$
$$b = d-c$$

$a+2d-2c=c \rightarrow a = 3c-2d$

★ $\begin{pmatrix}1&2\\1&3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix} \Longleftrightarrow \begin{pmatrix}3&-2\\-1&1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$

a

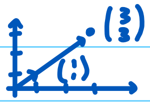
b

*span might not be $\mathbb{R}^2$...

→ could be the origin

→ could be a line

Ex: $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix}\right\} = \left\{\begin{pmatrix}a\\a\end{pmatrix} \middle| a \in \mathbb{R}\right\}$



$\begin{pmatrix}1\\0\end{pmatrix}$ not in span

$a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ has no soln.

(3D)

Ex: $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ span $= \mathbb{R}^3$

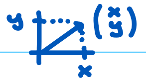
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

★ Ex: $\text{span}\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}3\\-1\\-2\end{pmatrix}\right\}$ $x+y+z=0$ is plane in $\mathbb{R}^3$

*if all 3 vect. in a plane through origin, implies linear combos also there

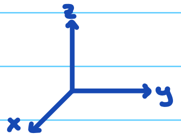
normal vectors is $\begin{pmatrix}1\\1\\1\end{pmatrix}$

4) LENGTH + DOT PRODUCT



$$\left.\begin{aligned} \text{pythag} \Rightarrow \left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 &= x^2+y^2 \\ \left\|\begin{pmatrix}x\\y\end{pmatrix}\right\| &= \sqrt{x^2+y^2} \end{aligned}\right\} \text{2D}$$

$\mathbb{R}^3$:



$$\left\|\begin{pmatrix}x\\y\\z\end{pmatrix}^2\right\| = \left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 + z^2 = x^2+y^2+z^2$$

$\mathbb{R}^n$: $\left\|\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\right\| = x_1^2 + x_2^2 + \dots + x_n^2 = \sum_{i=1}^{n} x_i^2$

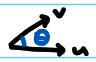
if $\|\vec{u}\| = 1$, $\vec{u}$ = unit vector

DOT PRODUCT:

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \cdot \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix} = \sum_{i=1}^{n} x_i y_i$$

$(\|\vec{u}\|^2 = \vec{u} \cdot \vec{u})$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ where θ = angle b/t



true in $\mathbb{R}^2$ → true in $\mathbb{R}^n$

$|\cos\theta| \leq 1 \Rightarrow |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ ineq.

$\cos\frac{\pi}{2} = 0 \Rightarrow \vec{u}, \vec{v} \perp$ when $\vec{u} \cdot \vec{v} = 0$

$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

TRIANGLE INEQ: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

9/5/25

vectors: add $\vec{u}+\vec{v}$

scalar mult. $c\vec{u}$

linear combos: $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$

LINEAR INDEPENDENCE / DEPENDENCE:

$\{\vec{v}_1, \dots, \vec{v}_k\}$ linearly dependent if (at least) one is a linear combo of others

opposite: linear dependent

Ex: in $\mathbb{R}^3$ $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}3\\4\\0\end{pmatrix} \right\}$ DEPENDENT!

$\begin{pmatrix}3\\4\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\\0\end{pmatrix}$

Ex: $\mathbb{R}^3$ $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ INDEPENDENT!

can't write 3rd or 2nd in terms of others

$\{\vec{v}_1, \dots, \vec{v}_k\}$ independent if anytime $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0}$,

we must have $a_1, a_2, \dots, a_k = 0$

not independent → make a_i's give $\vec{0}$ with some $a_j \neq 0$

$\rightarrow \vec{v}_j = -\sum_{i \neq j} (a_i/a_j)\vec{v}_i$

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$ & $a_1 \neq 0$

$a_1\vec{v}_1 = -a_2\vec{v}_2 - a_3\vec{v}_3 \rightarrow \vec{v}_1 = -(a_2/a_1)\vec{v}_2 - (a_3/a_1)\vec{v}_3$

MATRICES:

$A = \begin{pmatrix}1 & 5\\2 & 6\\3 & 7\\4 & 8\end{pmatrix}$ array of #s in rows + cols

— 4th row

2nd col

$A_{i,j}$ = entry in ith row, jth column

m×n matrix has m rows + n cols

column vector in $\mathbb{R}^k \longleftrightarrow k \times 1$ matrix

can think of matrix as collection of n col. vectors in $\mathbb{R}^m$

$$A = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right)$$

IDENTITY MATRIX: "do nothing"

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \dots, \ I_n =$$

DIAGONAL MATRIX:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

TRIANGULAR MATRIX:

$$\underset{\text{upper } \Delta}{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \underset{\text{lower } \Delta}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

RANK

$$\underset{3\times 4}{A} = \underset{\uparrow \ \ \uparrow \ \ \uparrow \ \ \uparrow \ \ 4 \text{ col vects}}{\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}}$$

column rank(A) = # linearly independent cols

working from left:

$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is lin. ind. b/c $a \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$ only if $a = 0$

$\Rightarrow \text{rank}(A) \geq 1$

$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ also ind. (look @ 2nd entries)

$\Rightarrow \text{rank}(A) \geq 2$

$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ also ind (look @ 2nd & 3rd entries)

$\Rightarrow \text{rank}(A) \geq 3$

solve $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \begin{pmatrix} -a_1 + a_2 \\ -a_2 + a_3 \\ a_3 \end{pmatrix} = \vec{0}$

$a_3 = 0 \Rightarrow a_2 = 0 \Rightarrow a_1 = 0$

* don't have to check 4th col. b/c vects. in $\mathbb{R}^3$ (3x4) highest rank is 3

$\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ not ind

note: $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{0}$

COLUMN SPACE $C(A)$**:** span of column vectors

✪ Ex: $C(A) = \mathbb{R}^3 \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -x\vec{v}_1 - y(\vec{v}_1 + \vec{v}_2) + z$

C(A) = range (or image) of mapping given by A via multiplication

= linear combo $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4$

Ex: $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -a+b \\ -b+c \\ -c+d \end{pmatrix}$ ← entries are dot products of row vecs with $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

mult. by A is a map from $\mathbb{R}^4$ to $\mathbb{R}^3$

RANK=1 matrices → any 2 column vectors = linearly dependent

Ex: $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{pmatrix}$

$\uparrow \uparrow \uparrow$ multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

row rank=1 also $2(1\ 3\ 4) = (2\ 6\ 8)$

$\begin{pmatrix} a & ca \\ b & cb \end{pmatrix}$ $\quad (b\ \ cb) = \frac{b}{a}(a\ \ ca)$

$\begin{pmatrix} a_1 & ba_1 & ca_1 & da_1 \\ a_2 & ba_2 & ca_2 & da_2 \\ a_3 & ba_3 & ca_3 & da_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}(1\ \ b\ \ c\ \ d)$ rows are mults. of $(1\ \ b\ \ cd)$

$3 \times 4 \qquad 3 \times 1 \qquad 1 \times 4$ ★

9/18/25: $m \times n$ matrix A $\Big(\quad\Big)\Big\}$ m rows $\vec{x} \in \mathbb{R}^n \to A\vec{x} \in \mathbb{R}^m$

$A\vec{x}$ in 2 ways:

a) $A\vec{x} = \begin{pmatrix} \text{row}_1(A)\cdot\vec{x} \\ \vdots \\ \text{row}_m(A)\cdot\vec{x} \end{pmatrix}$ b) $A\vec{x} = x_1 \text{col}_1(A) + \dots + x_n \text{col}_n(A)$

Ex: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ a) $\begin{pmatrix} x_1+3x_2 \\ 2x_1+4x_2 \end{pmatrix}$ b) $x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ★

ALGEBRA OF $A\vec{x}$

$A(c\vec{x}) = c\,A(\vec{x})$

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ } A is linear map

$A(c\vec{x}+d\vec{y}) = A(c\vec{x}) + A(d\vec{y})$

$= c\,A(\vec{x}) + d\,A(\vec{y})$

MATRIX MULTIPLICATION

A $m \times n$ $\quad$ AB is $m \times p$, is defined by $\text{col}_j(AB) = A\,\text{col}_j(B)$

B $n \times p$ $\quad A\begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ \vdots & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ \vdots & \vdots & & \vdots \end{pmatrix}$

ALGEBRA RULES:

a) $A(\overset{\text{scalar}}{c}B) = c(AB)$

b) $\overset{m\times n}{A}(\overset{n\times p}{B+C}) = AB + AC$ (distributive)

c) $\overset{m\times n}{A}(\overset{n\times p}{B}\overset{p\times q}{C}) = (AB)C$ (associative)

d) commutativity doesn't always hold, even for square matrices

(for multiplication)

Ex: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ not the same

EX: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ "squaring"

EX: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ for A 2×2 $\begin{cases} I_2 A = A \\ A I_2 = A \end{cases}$

3 ways to think ab. AB:

1) $(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$

✪ 2) $\text{col}_j(AB) = A\,\text{col}_j(B)$

3) $\text{row}_i(AB) = \underset{1 \times n}{\text{row}_i(A)}\ \underset{n \times p}{B}$

FACTOR $\underset{\uparrow}{\overset{m \times n}{A}} = \overset{m \times r}{C}\overset{r \times n}{R}$ $\quad r = \text{rank}(A)$

3a) rank = 1, $r = 1$ first

$A = \left(b_1\vec{a},\ b_2\vec{a},\ \dots,\ b_n\vec{a} \right)$ all cols. in same direction as $\vec{a}$

$A = \vec{a}\,(b_1, \dots, b_n) = \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}}_{\substack{\text{col.} \\ \text{vect.} \\ m \times 1 \\ C}} \underbrace{(b_1, \dots, b_n)}_{\substack{\text{row vect.} \\ 1 \times n \\ R}}$

3b) rank $= r > 1$

$\vec{c}_1, \dots, \vec{c}_r$ linearly independent cols. of A

$C = (\vec{c}_1, \vec{c}_2, \dots, \vec{c}_r) \in \mathbb{R}^{m \times r}$ ← $m \times r$

these coeffs give $\text{col}_j(R)$

every col. of A $\quad a_j$ = linear combo of $\vec{c}_1, \dots, \vec{c}_r$

✪ EX: $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$ $\quad \text{rank}(A) = 2$

1st 2 cols. are independent

$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ ← linearly ind. cols (could've chosen any 2 cols, then R is unique)

write cols of A as linear combo of C:

$\text{col}_3(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\text{col}_4(A) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

write R:

$R = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 2 \end{pmatrix}$ ← $\text{col}_1(A) = 1 \cdot \text{col}_1(C) + 0 \cdot \text{col}_2(C)$

top = what you want from 1st col of C to get result value in A
bot = what you want from 2nd col

$\underset{2 \times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \underset{2 \times 4}{\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 2 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$

* choice of C is NOT unique!
BUT R is unique conditional on C

NOTE: R contains $I_r \Rightarrow \text{rank}(R) \geq r$
$R \in \mathbb{R}^{r \times n} \Rightarrow \text{rank}(R) \leq r$ → $\text{rank}(R) = r$

R is reduced row echelon form of A

★ $\left\{ \begin{array}{l} \text{every column of } A \text{ is combination of cols. of } C, \ C \text{ has col. rank} = r \\ \text{every row of } A \text{ is combination of rows } R, \ R \text{ has row rank} = r \end{array} \right.$

square linear system

n eqns in n unknowns

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

define $A \in \mathbb{R}^{n\times n}$ with $A_{ij} = a_{ij}$

$\vec{x}_1 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ $A\vec{x} = \vec{b}$ is solvable <u>iff</u> $b \in C(A)$

9/10/25: $m \times n$ matrix is square if $m = n$

$\text{rank}(A) = \#$ lin. ind. cols

$= \#$ lin. ind. rows $\Rightarrow \text{rank}(A) \le \min\{m, n\}$

$A\vec{x}$ is linear combo of cols. of A

$A\vec{x} = x_1 \text{col}_1(A) + \dots + x_n \text{col}_n(A)$

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad &\vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n &= b_n \end{aligned} \iff \begin{array}{ll} A\vec{x} = \vec{b} & A_{i,j} = a_{ij} \\ \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \end{array}$$

Solveable $\iff \vec{b} \in C(A)$

✪ **SQUARE LINEAR SYSTEMS:**

thm: if $\text{rank}(A) = n$, then there exists a unique $\underset{n\times n \text{ matrix}}{A \in \mathbb{R}^{n\times n}}$ solution $\vec{x}$ to $A\vec{x} = \vec{b}$

for every $\vec{b}$

two claims:

a) there is a solution

$(A \mid \vec{b})$ has rank $= n \Rightarrow \vec{b}$ not indep. from 1st n cols $b \in C(A)$

b) there is <u>only one soln.</u>

assume there's 2 solns $A\vec{x} = \vec{b}$ & $A\vec{y} = \vec{b}$.

then $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y} = \vec{b} - \vec{b} = \vec{0}$

lemma: if A is $n\times n$, $\text{rank} = n$, and $A\vec{z} = \vec{0}$, then $\vec{z} = 0$

lemma $\Rightarrow \vec{x} - \vec{y} = \vec{0} \Rightarrow \vec{x} = \vec{y}$

why lemma? $z_1 \text{col}_1(A) + \dots + z_n \text{col}_n(A) = 0$ linear indep. cols $\Rightarrow z_1 + \dots + z_n = 0$

what happens if $\text{rank}(A) < n$?

either (a) no solns $\vec{b} \notin C(A)$

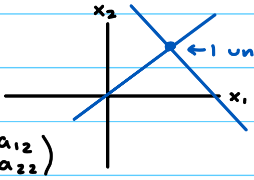
(b) ∞ solns $\vec{b} \in C(A)$ with ∞ ways to write

GEOMETRY IN $\mathbb{R}^2$:

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

2 lines in $\mathbb{R}^2$ $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$



← 1 unique soln

BACK SUBSTITUTION (upper Δ system)

$x + 2y = 1$ $\quad$ $x + y + z = 1$

$y = 1$ $\quad$ $y + 2z = 7$

$z = 9$

easy by working up

matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ ← upper Δs

Ex: $\left.\begin{matrix} x + y + z = 1 \\ z = 2 \\ z = 9 \end{matrix}\right]$ → no solns $\quad$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ "pivots" → any time a pivot = 0, smthg. has gone wrong

ELIMINATION: changing square system to upper Δ one

$\begin{cases} x + y = 1 \\ 2x + 3y = 0 \end{cases}$ → $-2(1\text{st}) + (2\text{nd})$

$-2x - 2y = -2$

$2x + 3y = 0$

$y = -2$

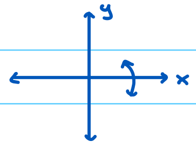
$x + y = 1$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is there a matrix E that does that

$E\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $E\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $E\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \text{col}_1(E) + x_2 \text{col}_2(E)$ (subtract 2×row₁ from row₂ & store into row₂)

$E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ $\quad$ $A\vec{x} = \vec{b}$ → $\underbrace{(EA)}_{\text{upper } \Delta}\vec{x} = E\vec{b}$

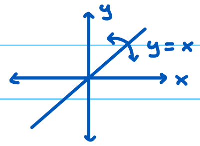
WRITING TRANSFORMATIONS AS MATRICES:



$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ -y \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

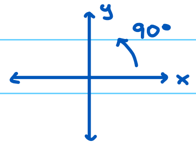
$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ → $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ → $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

9/12 review: for $A \in \mathbb{R}^{n \times n}$, $\operatorname{rank}(A) = \begin{matrix}\#\text{ ind cols}\\ \#\text{ ind rows}\end{matrix}$

$\operatorname{rank}(A) = n \Rightarrow C(A) = \mathbb{R}^n$ and $A\vec{x} = \vec{b}$ has unique soln for every $\vec{b}$

col. pic: $A\vec{x} = x_1 \text{col}_1(A) + \dots + x_n \text{col}_n(A)$

ELIMINATION (make matrix upper △)

two operations: (1) subtract multiple of row_i from row_j

(2) swap two rows

Ex: $A = \begin{pmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 6 & 8 & 4 \end{pmatrix} \xrightarrow[\text{add row 1 to r2}]{E_{21}} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & 8 & 4 \end{pmatrix} \xrightarrow[\text{sub. }2R_1\text{ from }R_3]{E_{31}} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 4 \end{pmatrix} \xrightarrow[R_3-3R_2]{E_{32}} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$ (upper △)

← all diagonals ≠ 0, none multiples ∴ rank = 3 !

all changes via matrix multiplication

what matrix E_{21} adds row_1 to row_2?

$E_{21}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

→ $E_{32}E_{31}E_{21}A = U$

INVERSE MATRIX (if it exists)

A is invertible if there is a "inverse matrix" A^{-1} so that $AA^{-1} = I = A^{-1}A$

note: not all A's have an A^{-1}

WHEN DOES A^{-1} EXIST? $\operatorname{rank}(A) = n \Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow A \in \mathbb{R}^n$ (col. space)

$\Leftrightarrow (A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0})$

A, B invertible ⇒ so is (AB) and $(AB)^{-1} = B^{-1}A^{-1}$

$B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

A = LU FACTORIZATION

→ can solve many $A\vec{x} = \vec{b}$ after finding U

$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{21}^{-1} =$ "subtract R_1 from R_2" $= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$, $E_{31}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$, $E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$

$E_{32}E_{31}E_{21}A = U \Rightarrow \overbrace{E_{32}^{-1}E_{32}}^{I}E_{31}E_{21}A = E_{32}^{-1}U$

mult. by E_{31}^{-1}: $E_{21}A = E_{31}^{-1}E_{32}^{-1}U$

mult. by E_{21}^{-1}: $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$A = LU$, where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

$E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ ← lower △ (have 1s on diag.)

COST OF ELIMINATION:

how many multiplications + subtractions needed to make $n \times n$ matrix upper Δ?

to eliminate 2,1 entry $\approx \underline{n}$ mult, $\underline{n}$ sub

$$\begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_n(A) \end{pmatrix}$$

1st col: $(n-1)$ entries to remove, each n mults, n subt.

2nd col: $(n-2)$ entries to remove, each $(n-1)$ mult, $(n-1)$ subt.

$\vdots$

$$\# \text{ total multiplications} = (n-1)n + (n-2)(n-1) + \ldots + 1(2)$$
$$= \sum_{j=1}^{n} j(j-1)$$
$$= \sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j \approx \frac{1}{3}n^3$$

idea: $\sum_{j=1}^{n} j^2 \approx \int_0^n x^2 dx = \frac{1}{3}n^3$ (Riemann's Sum)

9/15: for $A \in \mathbb{R}^{n \times n}$, $\text{rank}(A) = n \Rightarrow A^{-1}$ exists & $AA^{-1} = I = A^{-1}A$ elimination has 2 steps:

1) row operation
2) row exchange $\rightarrow$ upper Δ U

row reduction just matrix mult.

"cost" of $LU \sim \frac{1}{3}n^3$ mult, $\frac{1}{3}n^3$ subt

better to solve $A\vec{x} = \vec{b}$ w/ LU than A^{-1}

partial pivoting } $PA = LU$ standard for solving $A\vec{x} = \vec{b}$ computationally

$$\begin{tikzcd} \text{matlab/julia: "\textbackslash"} \arrow[dr] & \\ & \text{LA PACK} \\ \text{python: linalg.solve()} \arrow[ur] & \end{tikzcd}$$

TRANSPOSE: $A^T = A$

"reflect" across diagonal: $(A^T)_{ij} = A_{ji}$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 SPECIAL TYPES OF MATRICES:

symmetric: $A^T = A$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

skew-symmetric: $A^T = -A$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

note: $(AB)^T = B^T A^T$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \rightarrow ([AB]^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

$$(B^T A^T) = \sum_{k=1}^{n} B_{ik} A_{kj} \rightarrow \sum_{k=1}^{n} B_{ki} A_{jk}$$

same!

TRANSPOSE + DOT PRODUCT:

$\underbrace{\vec{x}, \vec{y}}_{n\times 1} \in \mathbb{R}^n \qquad \vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = \overset{1\times n}{(x_1, \cdots, x_n)} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^{n\times 1}$

$\vec{x}\cdot(A\vec{y}) = \vec{x}^T A \vec{y}, \quad (A\vec{x})^T\vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}\cdot(A^T\vec{y})$

*only square matrices have inverse

INVERSES: for $A \in \mathbb{R}^{n\times n}$, the following are equivalent:

1) $\mathrm{rank}(A) = n$
2) $A\vec{x} = \vec{b}$ solveable for any $\vec{b}$
3) $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

we know these already

4) there exists B s.t. $AB = I$
5) there exists C s.t. $CA = I$

new

· claim 1: $(5) \rightarrow ((1) = (2) + (3))$

we have C s.t. $CA = I$

if $A\vec{x} = \vec{0}$, then $CA\vec{x} = C\vec{0} \Rightarrow \vec{x} = \vec{0}$ ✓

· claim 2: $(4) \rightarrow ((1) = (2) = (3))$

have B s.t. $AB = I$

for any $\vec{b}$, $A(B\vec{b}) = (AB)\vec{b} = \vec{b} \Rightarrow B\vec{b}$ soln.

· claim 3: $((1) = (2) = (3)) \Rightarrow 4$

$I = (\vec{e}_1, \cdots, \vec{e}_n)$ solve $A\vec{y}_1 = \vec{e}_1, \cdots, A\vec{y}_n = \vec{e}_n$

$B = (\vec{y}_1, \cdots, \vec{y}_n) \Rightarrow AB = I$

· claim 4: $((1) = (2) = (3)) \Rightarrow 5$

need C s.t. $CA = I = (CA)^T = A^T C^T$ (rank = n — still when transpose rows & cols swap)

… same as claim 3!

more properties...

· if $AB = CA = I$, then $B = C$

$B = \overset{I}{(CA)}B = C\overset{I}{(AB)} = C$

· if AB has an inverse, so do A & B, and $(AB)^{-1} = B^{-1}A^{-1}$ (A & B = SQUARE)

A^{-1} should be $B(AB)^{-1}$: $AB(AB)^{-1} = I$

B^{-1} should be $A(BA)^{-1}$: $(AB)^{-1}AB = I$

$B^{-1}A^{-1} = \underbrace{(AB)^{-1}AB}_{I}(AB)^{-1} = (AB)^{-1}$

if A & B are square, and AB have rank n, (full rank)

then A & B have rank n!

PERMUTATION

Ex: how many perms of 1, 2, 3?

$\boxed{3! = 6}$

1, 2, 3	2, 1, 3	3, 1, 2
1, 3, 2	2, 3, 1	3, 2, 1

permutations of n elements: $n! = n(n-1)(n-2)\ldots 2 \cdot 1$

PERMUTATION MATRIX:

$$P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e_2^T \\ e_3^T \\ e_1^T \end{pmatrix} \qquad PI = P$$

simplest permutation: row exchange (swap 2 rows, rest stay in place)

THM: every permutation can be built out of exchanges

Ex: 1, 5, 3, 2, 4:

$$1, 2, 3, 4, 5 \xrightarrow[2 \leftrightarrow 5]{} 1, 5, 3, 4, 2 \xrightarrow[4 \leftrightarrow 5]{} 1, 5, 3, 2, 4$$

9/17 PROOF by INDUCTION on n = # elements (permutation matrices)

true for n=2: 1, 2 & 2, 1 (base case)

assume all permutations of 1, …, n can be built from row exchanges

consider permutation of 1, 2, …, n, n+1

2 CASES: if (n+1) is last, done!

if (n+1) is not last, make it last w/ an exchange

PROPERTIES OF PERMUTATION MATRICES

1) rank = n (invertible)
2) P^T also permutation matrix
3) rows of P are ⊥ to each other (cols too)
4) $P_1 P_2$ is also a permutation
5) $P^{-1} = P^T$ if rows are ⊥, $PP^T = I$ ← called orthogonal matrix (satisfies property)

$$\begin{pmatrix} \vec{p}_1 \\ \vdots \\ \vec{p}_n \end{pmatrix} \left(\vec{p}_1^{\,T} \ldots \vec{p}_n^{\,T} \right)$$

PA = LU FACT

A rank n ⇒ can reorder rows s.t. all pivots are non-zero during elimination

PA = LU where P = permutation matrix that puts rows of A in "right" order

PERMUTATIONS ON 3 ELEMENTS

Ex: 2, 3, 1 $\quad 1, 2, 3 \xrightarrow{\text{swap 1st 2}} 2, 1, 3 \xrightarrow{\text{swap last 2}} 2, 3, 1$

$$P_{231} = P_{132}\, P_{213} = \underset{132}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} \underset{213}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

=

*careful! pay attention to direction of multiplication

end of exam 1 content

TAYLOR APPROXIMATION

say $f(h) = O(g(h))$ if $|f(h)| \le C g(h)$ for h small constant C

$$f(x+h) = f(x) + hf'(x) + \underbrace{\tfrac{1}{2}h^2 f''(\xi)}_{O(h^2)}, \quad \xi \in [x, x+h]$$

$$f(x+h) = f(x) + h'f(x) + \tfrac{1}{2}h^2 f''(x) + \underbrace{\tfrac{1}{6}h^3 f^{(3)}(\xi)}_{O(h^3)}, \quad \xi \in [x, x+h]$$

FINITE DIFFERENCES

know f @ many points, how to approx. f' and f'' ?

$f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$ forward $O(h)$

$\approx \dfrac{f(x) - f(x-h)}{h}$ backward $O(h)$

$\approx \dfrac{f(x+h) - f(x-h)}{2h}$ centered $O(h^2)$

$$\begin{array}{rl} & f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + O(h^3) \\ - & f(x-h) = f(x) - hf'(x) + \tfrac{1}{2}h^2 f''(x) + \underbrace{O(h^3)} \\ \hline & f(x+h) - f(x-h) = 2hf'(x) + O(h^3) \end{array}$$

don't cancel b/c diff. #s

finding $f''(x)$:

$$\left.\begin{array}{rl} & f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \tfrac{1}{6}h^3 f'''(x) + O(h^3) \\ + & f(x-h) = f(x) - hf'(x) + \tfrac{1}{2}h^2 f''(x) - \tfrac{1}{6}h^3 f'''(x) + O(h^3) \\ \hline \end{array}\right] \text{cancelling out } f'(x)$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^3)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h) \leftarrow \text{accurate to } \sim h^2$$

9/22: $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$ forward diff. for f' (accurate to $\sim h$)

$f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ centered diff. for f'' (accurate to $\sim h^2$)

FINITE DIFFERENCE MATRICES

have $u(x)$ for $x \in [0, 1]$

discretize: look @ $u(j/N+1)$

define some vector $\vec{u} \in \mathbb{R}^{N+2}$ where $\overset{0 \text{ to } N+1}{\vec{u}_j = u(j/N+1)}$

$\vec{u}_0 = u(0)$, $\vec{u}_{N+1} = u(1)$, $h = 1/N+1$

FORWARD DIFF: "derivative" of $\vec{u}$ is $\vec{f} \in \mathbb{R}^{N+1}$ with $\vec{f}_j = \dfrac{\vec{u}_{j+1} - \vec{u}_j}{h}$ 0 to N

$\underset{(N+1)\times(N+2)}{A}\,\underset{N+2}{\vec{u}} = \vec{f} \leftarrow$ N+1 len vector

$$A = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{pmatrix} \longleftarrow \text{matrix encoding } \vec{f_j}$$

solns. to $A\vec{u} = \vec{0}$ are $\vec{u} = c\begin{pmatrix} \vdots \end{pmatrix}$ "what function's derivatives = 0?"

rank$(A) = N+1$

Ex: $3\times4 \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \vdots \end{pmatrix} = \vec{0}$

2nd DERIV: $\begin{cases} u''(x) = f(x) \text{ for } x \in (0,1) \\ u(0) = 0,\ u(1) = 0 \end{cases}$

set $\vec{f}_j = \frac{1}{h^2}\left(\vec{u}_{j-1} - 2\vec{u}_j + \vec{u}_{j+1}\right)$ for $j = 1, 2, \dots, N$

$= f(j/N+1)$ and set $\vec{u}_0 = 0$, $\vec{u}_{N+1} = 0$

$\Rightarrow \vec{f} = -\frac{1}{h^2} K \vec{u}$, where $K = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N \times N}$

$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$

Ex: solve $u''(x) = -1$, $x \in (0,1)$

$u(0) = u(1) = 0$ $\Rightarrow u(x) = \frac{x(1-x)}{2}$

$\rightarrow$ solve this, get & plot u $\rightarrow$ solving linear soln. gives us basically exact.

error is $\sim 10^{-16}$ $\rightarrow$ comes from taylor series. formula is exacl

Ex: $u''(x) = -2\pi x \sin(2\pi x)$

$u(0) = u(1) = 0$ #solve using LU factorization

lin. alg. good way to solve diff eqs.

ABSTRACT VECTOR SPACE set V of vectors... two operations:

(1) vector addition

(2) scalar multiplication

8 RULES:

Abelian group

1) commutivity $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

2) associativity

3) $\vec{0}$ exists

4) $\vec{x} + (-\vec{x}) = 0$

how vectors interact w/ scalars

5) $1 \cdot \vec{x} = \vec{x}$

6) $(ab)\vec{x} = a(b\vec{x})$

7) $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

8) $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

Exs of abstract vector space

- vectors in $\mathbb{R}^n$
- all $m \times n$ matrices $\rightarrow$ $A, B \in V$, $A + B \in V$ (matrices themselves can form vector space)
- polynomials on $\mathbb{R}$ $\rightarrow$ $x + (x^2 - 3) = x^2 + x - 3$ (starting to look like vector space)

Exs of NON-abstract vector spaces:

- $N \times N$ invertible matrices $\rightarrow$ $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ is NOT invertible $\Rightarrow$ not a vector space

also, adding 2 invertible (full rank) matrices does NOT guarantee full rank

multiplication does NOT exist b/t vectors

9/26

Ex: $$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_3 = 1 \\ x_2 + 2x_3 + x_4 = 1 \end{cases}$$ how many solns? 0

SUBSPACES:

set $S \subset V$ is a subspace if:

a) $\vec{v}, \vec{w} \in S \Rightarrow \vec{v} + \vec{w} \in S$

b) $\vec{v} \in S,\ a \in \mathbb{R} \Rightarrow a\vec{v} \in S$

Ex: line through $\vec{0}$ in $\mathbb{R}^n$:

$S = \{\vec{0}\}$ is subspace

$\{a > 0\} \subset \mathbb{R}$ not subspace

if S is a subspace, linear combinations stay in S.

Ex: $C(A)$ is all lin. combs of cols of $A = (\vec{a}_1, \ldots, \vec{a}_n)$ is subspace:

$\vec{v} = c_1\vec{a}_1 + \ldots + c_n\vec{a}_n$

$\vec{w} = d_1\vec{a}_1 + \ldots + d_n\vec{a}_n$

$\vec{v} + \vec{w} = (c_1 + d_1)\vec{a} + \ldots + (c_n + d_n)\vec{a}_n$

$\alpha\vec{v} = (\alpha c_1)\vec{a}_1 + \ldots + (\alpha c_n)\vec{a}_n$

$R(A)$ rowspace = all linear combos of rows of A

NULLSPACE:

$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

for m×n matrix A,
m eqns in $\mathbb{R}^n$

is a subspace:

$A\vec{v} = \vec{0},\ A\vec{w} = \vec{0} \Rightarrow A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{0} + \vec{0} = \vec{0} \Rightarrow \vec{v} + \vec{w} \in N(A)$

$A(\alpha\vec{v}) = \alpha(A\vec{v}) = \alpha\vec{0} = \vec{0} \Rightarrow \alpha\vec{v} \in N(A)$

two perspectives:

$A\vec{x} = x_1 \text{col}_1(A) + x_2 \text{col}_2(A) + \ldots + x_n \text{col}_n(A)$

$N(A)$ is way to make $\vec{0}$ out of cols of A

cols lin. indep $\Rightarrow N(A) = \{\vec{0}\}$ ← $N(A)$ = ways to make $\vec{0}$ out of cols in A

$$A\vec{x} = \begin{pmatrix} \text{row}_1(A) \cdot \vec{x} \\ \vdots \\ \text{row}_m(A) \cdot \vec{x} \end{pmatrix}$$ vectors in $N(A)$ are orthogonal (⊥) to all m rows in A

4TH FUNDAMENTAL SUBSPACE:

$N(A^T)$ = vectors ⊥ to all columns of A

★ Ex: $$R = \begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{pmatrix}$$ note: R is in rref (reduced row echelon form)

indep. cols. form I

dependent cols. called "free" vars.

$C(R) = \mathbb{R}_2$ ← first 2 cols of R are indep.

$N(R) = ?$ $R\vec{x} = \vec{0} \Leftrightarrow$ $$\begin{aligned} x_1 + 3x_3 + 5x_4 &= 0 \\ x_2 + 4x_3 + 6x_4 &= 0 \end{aligned}$$

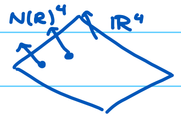
two special solns:

$$\left.\begin{aligned} x_3 = 1,\ x_4 = 0 &\rightarrow x_1 = -3,\ x_2 = -4 \\ x_3 = 0,\ x_4 = 1 &\rightarrow x_1 = -5,\ x_2 = -6 \end{aligned}\right] \Rightarrow \begin{pmatrix} -3 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -6 \\ 0 \\ 1 \end{pmatrix} \in N(R)$$

claim: $N(R) = \text{span}\{\vec{s}_1, \vec{s}_2\}$

1) they are independent

2) they are $\perp$ to rows of R



$\begin{pmatrix} R \\ \vec{s}_1^T \\ \vec{s}_2^T \end{pmatrix}^{4\times 4}$ rank = 4

if exists $\vec{s}_3 \in N(R)$ with $s_3 \notin \text{span}\{\vec{s}_1, \vec{s}_2\}$, then $\begin{pmatrix} R \\ s_1^T \\ s_2^T \\ s_3^T \end{pmatrix}$ would be rank 5 → NOT possible.

how to "see" special solns:

$R = (\underset{\text{identity}\uparrow}{I_2} \;\; \underset{\nwarrow \text{2x2 w/ depend. cols.}}{F})$ $\qquad (\vec{s}_1 \; \vec{s}_2) = \begin{pmatrix} -F \\ I_2 \end{pmatrix}$ $\qquad R(\vec{s}_1 \vec{s}_2) = \overset{R}{(I_2 F)} \begin{pmatrix} -F \\ I_2 \end{pmatrix}$

$= I_2 \cdot (-F) + F \cdot (I_2) = 0$

from A to rref(A) using elimination

3 operations to simplify a matrix:

a) subtract mult. of a row from another E_{ij}

b) row exchange P

new! → c) multiply a row by nonzero constant ← diagonal w/ m-1 ones, a single const ≠ 0 that gives scaling

$A \rightarrow \underset{\uparrow \text{invertible, prod. of ops}}{E} A =: R_0$

9/29: three operations to simplify matrix to rref:

a) subtract $c \cdot \text{row}_i$ from row_j

b) row exchange

c) multiply row by non-zero constant

each operation invertible: $A \rightarrow EA = R_0$

Ex: $\begin{pmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{indep. cols.}} & \underbrace{\begin{matrix} 3 & 5 \\ 4 & 6 \end{matrix}}_{\text{deped. cols}} \end{pmatrix}$

"free vars" → special solns.

from A to rref(A) with elimination:

R_0 has following properties:

a) 1st r indep. cols of R_0 will be $\overbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}^{r}$

(rank A)

b) last m-r rows all zero

c) remaining n-r cols are all free vars

(n-r) of these ⇒ get n-r special solns for N(A)

note: $N(A) = N(R_0)$, b/c $\underset{m\times n}{R_0} = \underset{\substack{m\times m \\ \text{invertible}}}{E} A$

$\vec{x} \in N(A) \Rightarrow R_0\vec{x} = E(A\vec{x}) = E\vec{0} = \vec{0}$ (everything in $N(A)$ is in $N(R_0)$

$\vec{x} \in N(R_0) \Rightarrow A\vec{x} = E^{-1}(EA\vec{x}) = E^{-1}(R_0\vec{x}) = E^{-1}\vec{0} = \vec{0}$ ✓ and vice versa)

Ex: $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{R_3-3R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{pmatrix}$

row 2 ÷3 $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{R_3-6R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ x_2 free var.

$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$R\vec{x} = \begin{pmatrix} x_1 + 2x_2 \\ x_3 \\ 0 \end{pmatrix} = \vec{0}$

for each free var x_2, x_3, etc,

get a special soln. by setting

one to 1, others to 0.

$x_2 = 1,\ x_3 = 0,\ x_5 = 0$ $\vec{s}_1$

$x_2 = 0,\ x_3 = 1,\ x_5 = 0$ $\vec{s}_2$

$\vdots$

SOLVING $\underset{m\times n}{A}\underset{\mathbb{R}^n}{\vec{x}} = \underset{\mathbb{R}^m}{\vec{b}}$ (theory) solveable $\Leftrightarrow \vec{b} \in C(A)$

if $m=n$ and A^{-1} exists, $\vec{x} = A^{-1}\vec{b}$ is unique soln.

if $\vec{y} \in N(A)$ and $A\vec{x} = \vec{b}$, then $A(\vec{x} + \alpha\vec{y}) = A\vec{x} + \alpha A\vec{y} = \vec{b} + \vec{0} = \vec{b}$

α scalar

*only way to find extra solns.

if $\vec{x}_1$ & $\vec{x}_2$ are solns. $A\vec{x}_1 = \vec{b}$ & $A\vec{x}_2 = \vec{b}$

$A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0} \Rightarrow \vec{x}_1 - \vec{x}_2 \in N(A)$

$A\vec{x} = \vec{b}$ in practice: $\underset{\substack{m\times m,\\ \text{invertible}}}{E}A = R_0 \leftarrow \text{rref}(A)$

$A\vec{x} = \vec{b} \Leftrightarrow EA\vec{x} = E\vec{b}$

$R_0\vec{x} = E\vec{b}$

$R_0 = \left(\begin{array}{c:c} I_r & F \\ \hdashline \multicolumn{2}{c}{0\text{'s}} \end{array}\right)\begin{matrix} \}r \\ \}m-r \end{matrix} \rightarrow R_0\vec{x} = E\vec{b}$ is solveable

$\Leftrightarrow$ last $m-r$ rows of $E\vec{b}$ are 0s

*once we have this form,
easy to tell solns

$C(R_0) = \{\vec{y} \in \mathbb{R}^m \mid \vec{y}_{r+1} = \ldots = \vec{y}_m = \vec{0}\}$

an obvious soln to $R\vec{x} = E\vec{b}$: set free vars $= 0$, and

"particular soln" $\vec{x}_p$ ↗ use the top r entries of $E\vec{b}$

$\Rightarrow$ general soln to $A\vec{x} = \vec{b}$ is $\vec{x}_p + \vec{x}_n$, where $\vec{x}_n \in N(A)$, $\vec{x}_n = c_1\vec{s}_1 + c_2\vec{s}_2 + \ldots + c_{n-r}\vec{s}_{n-r}$

note: don't compute E explicitly, apply it to $\vec{b}$.

Ex: $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

augmented matrix $(A \mid \vec{b}) = \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{cccc|c} 1 & \overset{x_2 \text{ free}}{3} & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 1 & 4 & b_3-b_1 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3-b_1-b_2 \end{array}\right)$

free vars.

I matrix

$R_0 = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$ need $b_3 - b_2 - b_1 = 0$ for soln

$E\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 - b_2 - b_1 \end{pmatrix}$

particular soln $\vec{x}_p = \begin{pmatrix} ?? \end{pmatrix}$

$x_2 = x_4 = 0$: $\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 - b_2 - b_1 \end{pmatrix} \rightarrow \boxed{\vec{x}_p = \begin{pmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{pmatrix}}$

10/1: last time... complete soln. to $A\vec{x}=\vec{b}$, $A=\begin{pmatrix}1&3&0&2\\0&0&1&4\\1&3&1&6\end{pmatrix}$, $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

aug. matrix $(A\,|\,\vec{b})=\left(\begin{array}{cccc|c}1&3&0&2&b_1\\0&0&1&4&b_2\\1&3&1&6&b_3\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&3&0&2&b_1\\0&0&1&4&b_2\\0&0&1&4&b_3-b_1\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&3&0&2&b_1\\0&0&1&4&b_2\\0&0&0&0&b_3-b_1-b_2\end{array}\right)$

I_2 x_2, x_4 free

$R_0=\begin{pmatrix}1&3&0&2\\0&0&1&4\\0&0&0&0\end{pmatrix}=\begin{pmatrix}R\\ \cdots\\ 0\text{'s}\end{pmatrix}$

particular soln: need $b_3-b_1-b_2=0$ (otherwise no soln)

"free"

set $x_2=x_4=0 \Rightarrow \vec{x}_p=\begin{pmatrix}b_1\\0\\b_2\\0\end{pmatrix}$

special solns: x_2 free $\rightarrow \vec{s}_1=\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}$ Complete soln: $\vec{x}_p+c_1\vec{s}_1+c_2\vec{s}_2$

x_4 free $\rightarrow \vec{s}_2=\begin{pmatrix}-2\\0\\-4\\1\end{pmatrix}$ $c_1, c_2 \in \mathbb{R}$

EX: $(A\,|\,\vec{b})=\left(\begin{array}{cccc|c}1&2&1&0&4\\2&4&4&8&2\\4&8&6&8&10\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&2&1&0&4\\0&0&2&8&-6\\0&0&2&8&-6\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&2&0&-4&7\\0&0&1&4&-3\\0&0&0&0&0\end{array}\right)$

R_0 $E\vec{b}$

$\left.\begin{array}{l}x_2=0\\x_4=0\end{array}\right]\rightarrow \vec{x}_p=\begin{pmatrix}7\\0\\-3\\0\end{pmatrix}$, x_2 free: $\vec{s}_1=\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}$

x_4 free: $\vec{s}_2=\begin{pmatrix}4\\0\\-4\\1\end{pmatrix}$ complete soln: $\vec{x}_p+c_1\vec{s}_1+c_2\vec{s}_2$

STANDARD BASIS: $\{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$

$\vec{e}_j=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$ ←jth comp.

$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=x_1\vec{e}_1+\dots+x_n\vec{e}_n$

- every $\vec{x}\in\text{span}\{\vec{e}_1,\dots,\vec{e}_n\}$
- only one way to do this

] = BASIS

Basis for vector space V:

list of vectors $\{\vec{v}_1, \vec{v}_2, \dots\}$ with

- span $\{\vec{v}_1, \vec{v}_2, \dots\}=V$
- $\vec{v}_1, \vec{v}_2, \dots$ are linearly independent

(b) ⇔ (b') each $\vec{v}\in V$ has unique linear combo. of $\{\vec{v}_1, \vec{v}_2, \dots\}$

$\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_k\vec{v}_k=d_1\vec{v}_1+d_2\vec{v}_2+\dots+d_k\vec{v}_k$

$\Rightarrow \vec{0}=(c_1-d_1)\vec{v}_1+\dots+(c_k-d_k)\vec{v}_k$

note: if i add/remove vector, no longer a basis!

remove one → span ≠ V

add one → linear dependence

EVERY BASIS: for $\mathbb{R}^n$ has n vectors in it

n+1 vectors → linear dependence $(\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n\ \vec{v}_{n+1})$ rank ≤ n

n−1 vectors → span ≠ $\mathbb{R}^n$ $\underset{n\times(n-1)}{(\vec{v}_1 \dots \vec{v}_{n-1})}\rightarrow R_0=\begin{pmatrix}\cdots ? \cdots\\ 0\text{'s}\end{pmatrix}$ ← $\vec{b}$ with $E\vec{b}$ non-zero in row not in span

DIMENSION:

thm: if $\{\vec{v}_1, \dots, \vec{v}_m\}$ and $\{\vec{w}_1, \dots, \vec{w}_n\}$ are basis for V, then

$m = n$. call this # dim(V).

proof: if $m > n$, $\vec{v}_k = a_{k1}\vec{w}_1 + \dots + a_{kn}\vec{w}_n$ for $k = 1, \dots, m$

$A = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{m \times n}$, rank $A < m \Rightarrow$ rows dependent

$c_1 \text{row}_1(A) + \dots + c_m \text{row}_m(A) = \vec{0}$

with c_i's not all 0.

$\Rightarrow c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0} \Rightarrow \{\vec{v}_1, \dots, \vec{v}_m\}$ lin. dep! ☐ ← if $m > n$, would all $= \vec{0} \rightarrow$ lin. dep.

POLYNOMIALS ON $\mathbb{R}$: is ∞-dimensional: $\{1, x, x^2, \dots\}$

polys of degree ≤ 3 on $\mathbb{R}$: $\{1, x, x^2, x^3\}$ is basis $\Rightarrow$ dim = 4

subspace $p(1) = 0$, dim = 3 $\{(x-1), (x-1)^2, (x-1)^3\}$ is basis

EX: V in $\mathbb{R}^3$ with $\left.\begin{matrix} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \end{matrix}\right]$ → looks like nullspace, which is a vector space.

$V = N(A)$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

rref: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ R_0

↑ x_2 free dim V=1

x_2 free $\rightarrow \vec{s}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

is a basis for $V = N(A)$

10/3 review: basis is list $\{\vec{v}_1, \vec{v}_2, \dots\}$ with: $\begin{cases} 1) \text{ span} = V \\ 2) \text{ linear independence} \end{cases}$

dim(V) = # elements in basis (independent of basis choice)

basis from list $\{\vec{v}_1, \dots, \vec{v}_k\}$ of vectors in V (finite dim)

a) if it spans, can delete some $\longrightarrow$ basis

do C of "CR fact". build some basis B going left to right. if $\vec{v}_i \cup B$ are lin. indep, add $\vec{v}_i$ to B

b) if lin indep, can add some $\longrightarrow$ basis

For $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ lin. indep. if span B = V, done!

otherwise, exists $\vec{w} \in V$ with $\vec{w} \notin$ span B.

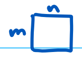
add $\vec{w}$ to B, and B still lin. indep.

repeat $n-k$ times for V with dim(V) = n.

B = basis

FUNDAMENTAL THM OF LIN. ALG:

for some $m \times n$ matrix A, rank(A) = r



dim C(A) = r   dim N(A) = n − r ← set of rows ⊥ to A

dim C(A^T) = r   dim N(A^T) = m − r

↑ set of all cols ⊥ to A

*we already know dim C(A) = r

indep. cols of A are basis for C(A)

← cols of RREF(A) w/ pivots

R_0 = rref(A)

$R_0 = \begin{pmatrix} I_r & \vdots & F \\ \cdots & & \cdots \\ & 0\text{'s} & \end{pmatrix}$

$R_0\vec{x} = \vec{0}$, $\vec{x} = \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} \begin{matrix}] \text{ indep.} \\ ] \text{ free} \end{matrix}$ r dim.

$\vec{y} = -F\vec{z}$

$R_0\vec{x} = \begin{pmatrix} I_r \\ \cdots \\ 0\text{s} \end{pmatrix}\vec{y} + \begin{pmatrix} F \\ \cdots \\ 0\text{'s} \end{pmatrix}\vec{z} = \begin{pmatrix} \vec{y} \\ \cdots \\ 0\text{'s} \end{pmatrix} + \begin{pmatrix} F\vec{z} \\ \cdots \\ 0\text{'s} \end{pmatrix} = \vec{0}$

N(A) = N(R_0) special solns $\{\vec{s}_1, \dots, \vec{s}_{n-r}\}$ basis for N(A)

*choose any z, we can pick y &

get unique vector

$N(A) = N(R_0)$?

$A \to R_0 = \overset{\text{invertible}}{E}A$ (through linear transformations)

if $\vec{x} \in N(A)$, $R_0\vec{x} = E(A\vec{x}) = \vec{0}$

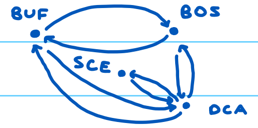
$\vec{x} \in N(R_0)$, $A\vec{x} = E^{-1}(EA)\vec{x} = E^{-1}(R_0\vec{x}) = E^{-1}\vec{0} = \vec{0}$

GRAPHS & INCIDENCE MATRIX:

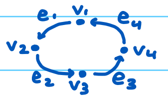
graph: collection of nodes (vertices) & edges that connect pairs

V = vertices, E = edges; graph is directed if edges have start + end

flights:



CYCLIC GRAPH on 4 nodes:



INCIDENCE MATRIX: describes graph

- rows: edges
- columns: vertices

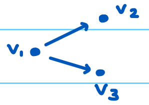
$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline e_1 & -1 & 1 & 0 & 0 \\ e_2 & 0 & -1 & 1 & 0 \\ e_3 & 0 & 0 & -1 & 1 \\ e_4 & 1 & 0 & 0 & -1 \end{array}$$

-1 e_i leaves v_j

$+1$ e_j goes to v_j

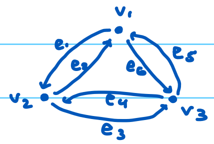
0 otherwise

TREE:



$$\begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline e_1 & -1 & 1 & 0 \\ e_2 & -1 & 0 & 1 \end{array}$$

COMPLETE GRAPH:



$$\begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline e_1 & -1 & 1 & 0 \\ e_2 & 1 & -1 & 0 \\ e_3 & 0 & -1 & 1 \\ e_4 & 0 & 1 & -1 \\ e_5 & 1 & 0 & -1 \\ e_6 & -1 & 0 & 1 \end{array}$$

$N(A) = \left\{ c\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \middle| c \in \mathbb{R} \right\}$ always if graph is connected

$\dim(N(A)) = 1 \Rightarrow d(C(A)) = n-1$

$n = |V|$, $m = |E|$ $\quad d(C(A^T)) = n-1$

$d(N(A^T)) = m(n-1)$

10/6 **REVIEW: FTLA** for $m \times n$ A, $\text{rank}(A) = r$

$\dim(C(A)) = r \qquad \dim(N(A)) = n-r$

$\dim(C(A^T)) = r \qquad \dim(N(A^T)) = m-r$

ORTHOGONALITY: $\perp$

subspaces V & W are $\perp$ if $\vec{v} \cdot \vec{w} = 0$ for all $\vec{v} \in V$, $\vec{w} \in W$

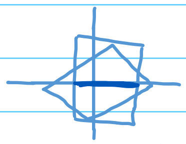
Ex: $V = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \right\}$, $W = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right\}$ $\perp$

dim=2 dim=1

$V = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \right\}$, $W = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \right\}$ $\perp$

dim=1 dim=1

2 lines!



orthogonal but NOT orthogonal complements → $V = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \right\}$ $W = \left\{ \begin{pmatrix} 0 \\ s \\ t \end{pmatrix} \right\}$ not $\perp$ ← some might be, but not all

CLAIM: if $V \perp W$ and $\vec{x} \in V \cap W$, then $\vec{x} = \vec{0}$

$\vec{x} \cdot \vec{x} = 0 \rightarrow ||\vec{x}||^2 = 0 \rightarrow \vec{x} = \vec{0}$

for $m \times n$ matrix A, rows of $A \perp N(A)$

$\Rightarrow C(AT) \perp N(A)$

pf: if $\vec{v} \in C(A^T)$, $\vec{v} = c_1 \cdot \vec{r}_1 + c_2 \cdot \vec{r}_2 \ldots + C_m \vec{r}_m$, $r_i = \text{row}_i(A)$

for $\vec{x} \in N(A)$, $\vec{r}_i \cdot \vec{x} = 0$ for all $i = 1, \ldots, m$

$\Rightarrow \vec{v} \cdot \vec{x} = c_1(\vec{r}_1 \cdot \vec{x}) + c_2(\vec{r}_2 \cdot \vec{x}) + \ldots + c_m(\vec{r}_m \cdot \vec{x}) = 0 \quad \square$

FTLA pt 2: subspace V & W are orthogonal complements if

every vector ⊥ to V is in W.

$V^\perp$ is orthogonal complement of V, is $\overbrace{\text{largest subspace}}^{\text{largest dim, every } \perp \text{ matrix is subspace}}$ ⊥ V.

fact: for $V \perp W$ in $\mathbb{R}^n$, $\dim V + \dim W \leq n$

if $\dim V + \dim W = n$, $V^\perp = W$ & $W^\perp = V$ (orthog. comp.)

if $V \perp W$, with $\{\vec{v}_1, \ldots, \vec{v}_p\}$ basis for V

$\{\vec{w}_1, \ldots, \vec{w}_q\}$ basis for W

then $\{\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q\}$ are lin. indep. vectors in $\mathbb{R}^n$

if $p + q = n \Rightarrow$ basis for $\mathbb{R}^n$

pf: if $\vec{v} + \vec{w} = \vec{0}$ for some $\vec{v} \in V$, $\vec{w} \in W$

$||\vec{v} + \vec{w}||^2 = \vec{v} \cdot \vec{v} + \overbrace{\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v}}^{0} + \vec{w} \cdot \vec{w} = ||\vec{v}||^2 + ||\vec{w}||^2$

$\Rightarrow \vec{v} = \vec{0}$ & $\vec{w} = \vec{0} \quad \square$

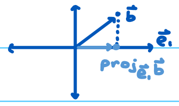
FTLA part 2: $\overset{\text{span of rows of A}}{C(A^T)^\perp} = N(A)$ and $C(A)^\perp = N(A^T)$

important fact: $N(\underbrace{A^T}_{n \times m} A) = N(\underbrace{A}_{m \times n})$

$A\vec{x} = \vec{0} \Rightarrow A^T A \vec{x} = \vec{0}$

$A^T A \vec{x} = \vec{0} \Rightarrow \underbrace{\vec{x}^T A^T A \vec{x}}_{(A\vec{x})^T (A\vec{x})} = \vec{0} \Rightarrow ||A\vec{x}||^2 = \vec{0} \Rightarrow A\vec{x} = \vec{0}$

PROJECTION: closest pt. on x-axis to $\vec{b}$



1802: proj. of $\vec{b}$ onto line through $\vec{a}$: $\vec{p} = \vec{a}\left(\frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\right)$

for $A \in \mathbb{R}^{m \times m}$ with $\overset{\text{full rank}}{\text{rank}(A) = n}$, how do you project some $\vec{x} \in \mathbb{R}^m$ onto $C(A)$?

CLAIM: $\overset{\text{proj}}{\vec{p}} = P\vec{x}$ where $P = A(A^TA)^{-1}A^T$

why is A^TA invertible? $N(\underbrace{A^TA}_{n \times n}) = N(A) = \{\vec{0}\}$

why does it work?

need to check $(P\vec{x} - \vec{x}) \perp C(A)$

$\Leftrightarrow (P\vec{x} - \vec{x}) \cdot (A\vec{y}) = 0$ for all $\vec{x} \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$y^TA^T(P\vec{x} - \vec{x}) = \vec{y}^TA^T(A(A^TA)^{-1}A^T\vec{x} - \vec{x})$

$= \vec{y}^TA^TA(A^TA)^{-1}A^T\vec{x} - \vec{y}^TA^T\vec{x}$

$= \vec{y}^TA^T\vec{x} - y^TA^T\vec{x} = 0$

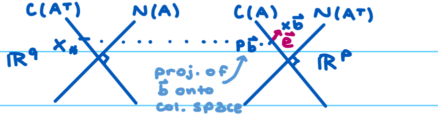
10/8 **LEAST SQUARES:**

1) eqns. w/o solns

2) fitting line into 3 pts

3) fitting in general

$A\vec{x} = \vec{b}$ $A \in \mathbb{R}^{p \times q}$, $\vec{x} \in \mathbb{R}^q$, $\vec{b} \in \mathbb{R}^p$



if $\vec{b} \in C(A)$, then there is a soln.

$\vec{b} = \underbrace{P\vec{b}}_{\in C(A)} + \underbrace{\vec{e}}_{\in N(A)}$ $P\vec{b}$ minimizes the distance to $\vec{b}$ on $C(A)$

QUESTION: what is the best approximation x^* for a soln to $A\vec{x} = \vec{b}$

solve $A\vec{x}_* = \underbrace{P\vec{b}}_{\in C(A)}$ instead. we know there is a soln. b/c $P\vec{b} \in C(A)$

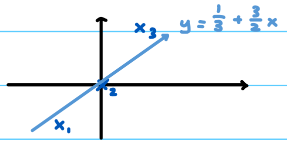
CLAIM: $\vec{x}_*$ solves $A^TA\vec{x}_* = A^T\vec{b}$

why? $A\vec{x}_* = P\vec{b} = \underbrace{\vec{b} - \vec{e}}_{\in N(AT)}$

$A^TA\vec{x} = A^T\vec{b} + \underbrace{A^T(-\vec{e})}_{0 \text{ since } \vec{e} \in N(AT)}$

FITTING LINE INTO 3 POINTS:

$(\underset{x_1}{-1}, \underset{y_1}{-1})$, $(\underset{x_2}{0}, \underset{y_2}{0})$, $(\underset{x_3}{1}, \underset{y_3}{2})$



what is line $y = c + dx$ that fits the data $(x_i, y_i)_{i \in \{1,2,3\}}$ the best?

ACTUAL SOLN would be: $c + dx_i = y_i$

$$\begin{cases} c + d \times (-1) = -1 \\ c + d \times (0) = 0 \\ c + d \times (1) = 2 \end{cases}$$

NOT CONSISTENT! $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ $A\vec{x} = \vec{b}$

the best $\vec{x}_* = \begin{pmatrix} c_* \\ d_* \end{pmatrix}$ solves ⊛: $A^TA\vec{x}_* = A^T\vec{b}$

$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ ← invertible! almost always

$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

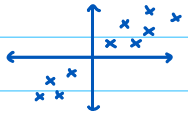
$$\begin{cases} 3c_* = 1 & \rightarrow c_* = 1/3 \\ 2d_* = 3 & \rightarrow d_* = 3/2 \end{cases}$$

∴ best fitting line is $y = \frac{1}{3} + \frac{3}{2}x$

the solution (c_*, d_*) minimizes $\sum_{i=1}^{3} (c + dx_i - y_i)$ over $(c, d) \in \mathbb{R}^2$

hence the name "least squares approximation"

FITTING IN GENERAL:



data points $(a_i, b_i)_{i \in \{1, \dots, m\}}$

best line $y = c + dx$ fitting the data?

$c + da_i = b_i$, m such lines

$$\Leftrightarrow \underset{A}{\begin{pmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_m \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} c \\ d \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

BEST-FITTING LINE solves $A^TA\vec{x}_* = A^T\vec{b}$

$A^TA = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_m \end{pmatrix}\begin{pmatrix} 1 & a_1 \\ \vdots & \vdots \\ 1 & a_m \end{pmatrix} = \begin{pmatrix} m & \Sigma a_i \\ \Sigma a_i & \Sigma a_i^2 \end{pmatrix}$, $A^T\vec{b} = \begin{pmatrix} \Sigma b_i \\ \Sigma a_i b_i \end{pmatrix}$

↳ invertible iff $m\Sigma a_i^2 \neq (\Sigma a_i)^2$

$(A^TA)^{-1} = \frac{1}{m\Sigma a_i^2 - (\Sigma a_i^2)}\begin{pmatrix} \Sigma a_i^2 & -\Sigma a_i \\ -\Sigma a_i & m \end{pmatrix}$

solution $\vec{x}_*$ is given by $(A^TA)^{-1}\begin{pmatrix} \Sigma b_i \\ \Sigma a_i b_i \end{pmatrix}$

NOTE: much simpler if $\Sigma a_i = 0$ → "centered data"

10/10: **ORTHOGONAL VECTORS:**

kronecker $\delta = \delta_{ij}$ is (i,j)th entry of I

$\{\vec{v}_1, \dots, \vec{v}_k\}$ orthonormal if $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$ for all i, j

$Q = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \in \mathbb{R}^{n \times k}$

$Q^TQ = I_n$ $\quad$ $QQ^T \neq I_n$ unless $k = n$

$(Q^TQ)_{ij} = \vec{v}_i \cdot \vec{v}_j$ $\quad$ $\operatorname{rank}(QQ^T) = \dim(C(QQ^T)) \leq \dim(C(Q)) \leq k$

proj. onto cols of Q $(C(Q))$:

$Q(Q^TQ)^{-1}Q^T = QI^{-1}Q^T = QQ^T$

least squares for $Q\vec{x} = \vec{b}$: $\quad Q^TQ\vec{x} = Q^T\vec{b} \Rightarrow \vec{x} = Q^T\vec{b}$

ORTHOGONAL MATRICES in $\mathbb{R}^n$.

$n \times n$ matrix $Q = (\underbrace{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n}_{})$ orthogonal vectors

Ex: in $(\vec{e}_1, \vec{e}_2, \dots \vec{e}_n)$ orthogonal

Q orthogonal → $\underset{n \times n}{Q^T} \underset{n \times n}{Q} = I_n \rightarrow Q^T = Q^{-1}$

$QQ^T = I_n$ also → Q^T also orthogonal

other examples:

a) permutation $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

b) rotation $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

c) reflections $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$

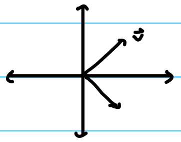
general reflection across some hyperplane:

$V^{\perp}$: $Q = I - 2\vec{v}\vec{v}^T$ for $\|\vec{v}\| = 1$

for $\vec{v}$, need matrix R s.t.

1) $R\vec{v} = -\vec{v}$ & 2) $R\vec{x} = \vec{x}$ for all $\vec{x} \cdot \vec{v} = 0$

$(I - 2\vec{v}\vec{v}^T)\vec{v} = \vec{v} - 2\vec{v}(\vec{v}^T\vec{v}) = -\vec{v}$ | $(I - 2\vec{v}\vec{v}^T)\vec{x} = \vec{x} - 2\vec{v}(\overset{0}{v^T\vec{x}}) = \vec{x}$



$I - \vec{v}\vec{v}^T$ sends to $\vec{v}^{\perp}$
$\vec{x}_{proj} = (I - \vec{v}\vec{v}^T)\vec{x}$

to reflect:
subtract another $\vec{v}\vec{v}^T\vec{x}$

claim: orthogonal matrices preserve lengths & angles

$\|Q\vec{x}\| = (Q\vec{x}) \cdot (Q\vec{x}) = \vec{x}^T \overbrace{Q^TQ}^{I} \vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$

$(Q\vec{x}) \cdot (Q\vec{y}) = (Q\vec{y})^T(Q\vec{x}) = \vec{y}^T Q^T Q \vec{x} = \vec{y}^T\vec{x} = \vec{x} \cdot \vec{y}$

GRAM-SCHMIDT ORTHOGONALIZATION

$\{\vec{v}_i, \dots, \vec{v}_k\} \rightarrow \{\vec{w}_1, \dots, \vec{w}_k\} \rightarrow \{\vec{q}_1, \dots \vec{q}_k\}$
any basis orthog. on vectors

1) $\vec{w}_1 = \vec{v}_1$

✪ 2) $\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1} \vec{v}_2 = \vec{v}_2 - \vec{w}_1 \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1}$

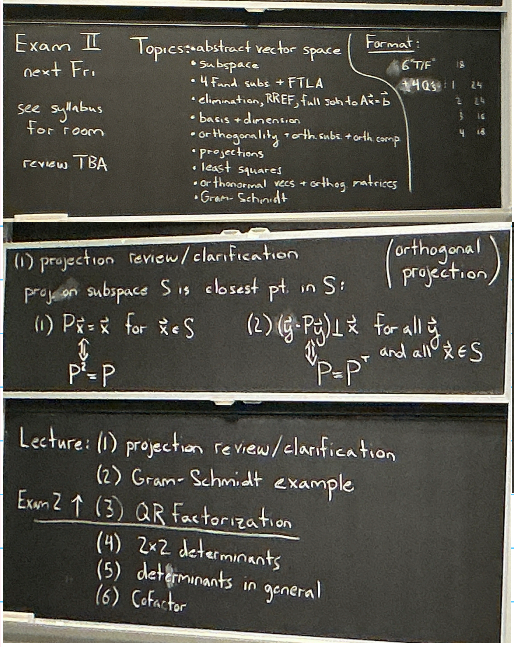
note: $\vec{w}_1 \cdot \vec{w}_2 = \vec{w}_1 \left(\vec{v}_2 - \vec{w}_1 \frac{(\vec{w}_1 \cdot \vec{v}_2)}{\vec{w}_1 \cdot \vec{w}_1}\right. = \vec{w}_1 \cdot \vec{v}_2 - \vec{w}_1 \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1} = 0$

3) $\vec{w}_3 = \vec{v}_3 - \text{proj}_{\text{span}\{w_1, w_2\}} \vec{v}_3 = \vec{v}_3 - \frac{\vec{w}_1 (\vec{w}_1 \vec{v}_3)}{\vec{w}_1 \cdot \vec{w}_1} - \frac{\vec{w}_2 (\vec{w}_2 \vec{v}_3)}{\vec{w}_2 \cdot \vec{w}_2}$

↳ can decouple b/c w_1 & w_2 are orthog.

last step: $\vec{q}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|}$ for $i = 1, \dots, k$

10/15



Exam II
next Fri

see syllabus
for room

review TBA

Topics: • abstract vector space
- subspace
- 4 fund. subs + FTLA
- elimination, RREF, full soln to $A\vec{x}=\vec{b}$
- basis + dimension
- orthogonality + orth. subs. + orth. comp
- projections
- least squares
- orthonormal vecs + orthog. matrices
- Gram-Schmidt

Format:

6 "T/F"		18
+ 4 Q's:	1	24
	2	24
	3	16
	4	18

(1) projection review/clarification (orthogonal projection)

proj. on subspace S is closest pt. in S:

(1) $P\vec{x}=\vec{x}$ for $\vec{x}\in S$ $\Updownarrow$ $P^2=P$

(2) $(\vec{y}-P\vec{y})\perp\vec{x}$ for all $\vec{y}$ and all $\vec{x}\in S$ $\Updownarrow$ $P=P^T$

Lecture: (1) projection review/clarification

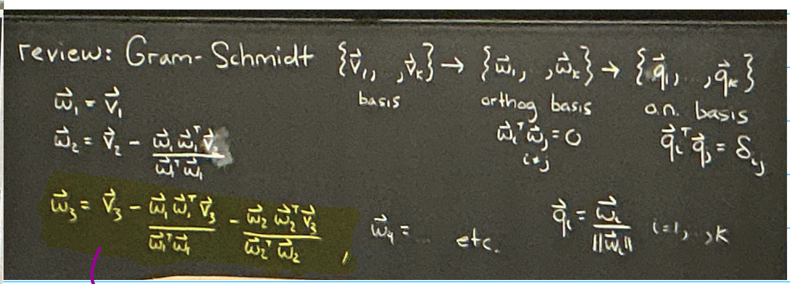
(2) Gram-Schmidt example

Exam 2 ↑ (3) QR Factorization

(4) 2x2 determinants

(5) determinants in general

(6) Cofactor



review: Gram-Schmidt $\{\vec{v}_1, \ \ , \vec{v}_k\} \to \{\vec{w}_1, \ \ ,\vec{w}_k\} \to \{\vec{q}_1, \ \ ,\vec{q}_k\}$

basis — orthog basis $\vec{w}_i^T\vec{w}_j = 0$, $i\neq j$ — o.n. basis $\vec{q}_i^T\vec{q}_j=\delta_{ij}$

$\vec{w}_1=\vec{v}_1$

$\vec{w}_2=\vec{v}_2-\frac{\vec{w}_1\vec{w}_1^T\vec{v}_2}{\vec{w}_1^T\vec{w}_1}$

$\vec{w}_3=\vec{v}_3-\frac{\vec{w}_1\vec{w}_1^T\vec{v}_3}{\vec{w}_1^T\vec{w}_1}-\frac{\vec{w}_2\vec{w}_2^T\vec{v}_3}{\vec{w}_2^T\vec{w}_2}$, $\vec{w}_4 = \ldots$ etc.

$\vec{q}_i=\frac{\vec{w}_i}{\|\vec{w}_i\|}$ $i=1,\ldots,k$

build basis iteratively

PROJECT onto span of cols of A (w/ ind. cols): $A(A^TA)^{-1}A^T$

PROJECT on subspace S is closest pt in S:

X 1) $P\vec{x}=\vec{x}$ for $\vec{x}\in S$

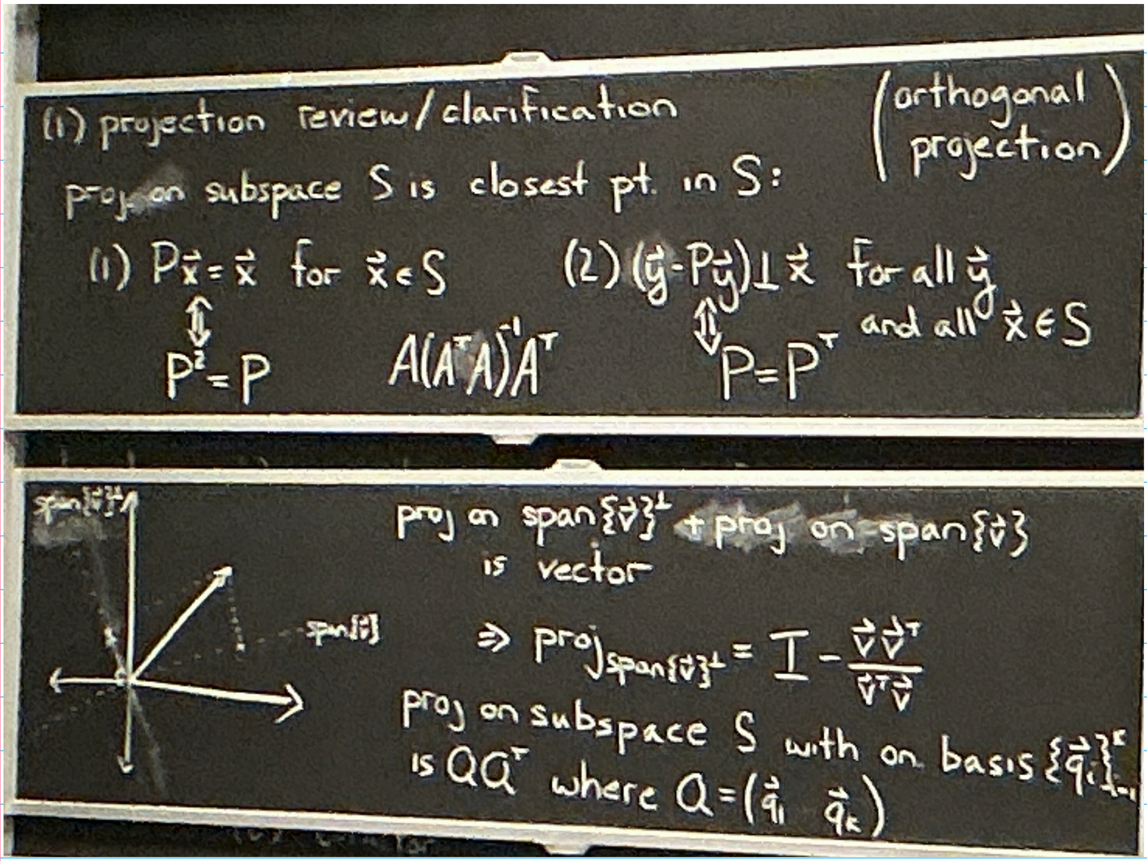
$\downarrow$

$P^2=P$

orthog. proj. [2) $(\vec{y}-P\vec{y})\perp\vec{x}$ for all $\vec{y}$ & all $\vec{x}\in S$

$\updownarrow$

$P=P^T$



(1) projection review/clarification (orthogonal projection)

proj. on subspace S is closest pt. in S:

(1) $P\vec{x}=\vec{x}$ for $\vec{x}\in S$ $\Updownarrow$ $P^2=P$

$A(A^TA)^{-1}A^T$

(2) $(\vec{y}-P\vec{y})\perp\vec{x}$ for all $\vec{y}$ and all $\vec{x}\in S$ $\Updownarrow$ $P=P^T$

proj on span$\{\vec{v}\}^\perp$ + proj on span$\{\vec{v}\}$ is vector

$\Rightarrow \text{proj}_{\text{span}\{\vec{v}\}^\perp} = I-\frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}$

proj on subspace S with o.n. basis $\{\vec{q}_i\}_{i=1}^k$ is QQ^T where $Q=(\vec{q}_1 \ \ \vec{q}_k)$

proj on span$\{\vec{v}\}^\perp$ + proj on span$\{\vec{v}\}$ is vector

$\Rightarrow \text{proj}_{\text{span}\{\vec{v}\}^\perp} = I-\frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}$

GRAM-SCHMIDT:

Ex: find orthonormal basis for $x+y+z=0$

is a subspace! $A=(1,1,1)$ $\overset{\shortparallel}{N(A)}$

$\text{rank}(A)=1$, FTLA: $\dim(N(A)) = 3-1 = 2$

y, z = free vars $\Rightarrow$ special solns $\vec{s}_1 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\vec{s}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

$\left\{\underset{\vec{v}_1}{\begin{pmatrix}-1\\1\\0\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}-1\\0\\1\end{pmatrix}}\right\}$ is basis for $N(A)$

GRAM-SCHMIDT: $\vec{w}_1 \cdot \vec{w}_1 = \vec{w}_1^T \vec{w}_1$ matrix · vect (proj v_2 onto span of w_1)

$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\vec{w}_2 = \vec{v}_2 - \vec{w}_1 \dfrac{\vec{w}_1^T \vec{v}_2}{\vec{w}_1^T \vec{w}_1} = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \left(\frac{1}{2}\right)\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix}$

orthonorm. basis: $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix}\right\}$ ← unit vect

QR FACTORIZATION: can use gram-schmidt

$\left[Q\begin{pmatrix}\pm1 & & 0\\ & \ddots & \\ 0 & & \pm1\end{pmatrix}\right]\left[\begin{pmatrix}\pm1 & & 0\\ & \ddots & \\ 0 & & \pm1\end{pmatrix}R\right]$

for nxn matrix A, can write $A = \underset{\substack{\uparrow\\ \text{orthog. matrix}}}{\overset{n\times n}{Q}}\underset{\substack{\nwarrow\\ \text{upper } \Delta}}{\overset{n\times n}{R}}$

$A\vec{x} = \vec{b} \Leftrightarrow QR\vec{x} = \vec{b}$ $\cdot Q^T$

$\Leftrightarrow \underset{I \text{ b/c O.R.}}{(Q^TQ)}R\vec{x} = Q^T\vec{b} \Leftrightarrow R\vec{x} = Q^T\vec{b}$

2x2 DETERMINANTS : # you can assign to square matrices

- tells us about volume, whether there's an inverse, etc.

$\det\begin{pmatrix}a & b\\c & d\end{pmatrix} = ad-bc$

PROPERTIES:

- $\det I_2 = 1$
- row exchange $\Rightarrow$ det mult. by (-1)
- $\det\begin{pmatrix}a & b\\c & d\end{pmatrix} = -\det\begin{pmatrix}c & d\\a & b\end{pmatrix}$
- $\det(A) = \det(A^T)$
- det is linear in each row:

linearity →
  a) $\det\begin{pmatrix}sa & sb\\sc & sd\end{pmatrix} = s\det\begin{pmatrix}a & b\\c & d\end{pmatrix}$
  b) $\det\begin{pmatrix}a+e & b+f\\c & d\end{pmatrix} = (a+e)d-(b+f)c = (ad-bc)+(ed-fc)$
  $= \det\begin{pmatrix}a & b\\c & d\end{pmatrix} + \det\begin{pmatrix}e & f\\c & d\end{pmatrix}$

$\det(A) \neq 0 \Rightarrow A^{-1}$ exists, $A^{-1} = \dfrac{1}{\det(A)}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$

verify: $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}d & -b\\-c & a\end{pmatrix} = \begin{pmatrix}ad-bc & 0\\0 & ad-bc\end{pmatrix} = \det(A)I_2$

$\det(A) = 0 \Rightarrow A^{-1}$ doesn't exist

$ad-bc=0 \Leftrightarrow ad=bc \Leftrightarrow \begin{pmatrix}a\\c\end{pmatrix}\begin{pmatrix}b\\d\end{pmatrix}$ have same slope

DETERMINANTS in general: define inductively, expand 1st row

$$\det\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\det\begin{pmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{pmatrix} \underset{\uparrow\text{alt. sign}}{-} a_{12}\det\begin{pmatrix}a_{21} & a_{23}\\a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{pmatrix}$$

EXAM 2 CONTENT ↑

10/17 **DETERMINANTS:** $|\cdot|$ means $\det(\cdot)$

arbitrary n:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = +a_{11}\begin{vmatrix} \text{cross out} \\ \text{1st row,} \\ \text{1st col} \end{vmatrix} - a_{12}\begin{vmatrix} \text{cross out} \\ \text{1st row,} \\ \text{2nd col} \end{vmatrix} + \dots$$

Ex:

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = +2\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 0 + 0$$

$$= 2\left(2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}\right) + \left(-1\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}\right)$$

$$= 2(2\cdot 3 + (-2)) + (-1\cdot 3 + 0) = 5$$

Co FACTORS:

$$C_{ij} = (-1)^{i+j}\begin{vmatrix} (n-1)\times(n-1) \text{ matrix} \\ \text{with row } i, \\ \text{col } j \text{ deleted} \end{vmatrix}$$

cofactor matrix $C \in \mathbb{R}^{n\times n}$

satisfies $AC^T = (\det A)I_n$

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c_{11} = d$ $c_{12} = -c$ $c_{21} = -b$ $c_{22} = a$ $\Rightarrow C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} C^T$$

in general, if A^T exists, $A^{-1} = \frac{1}{\det A} C^T$

PROPERTIES OF nxn DET:

U upper $\Delta \Rightarrow \det U =$ product of diagonal elements

*can immediately find det by multiplying diagonals

Ex: $\begin{vmatrix} 1 & 20^{10} & -\pi \\ 0 & 4 & 154.9 \\ 0 & 0 & 2 \end{vmatrix} = 1\cdot 4\cdot 2 = 8$

proof by induction:

base case: $n=1$ $\det(a) = a$ ✓

inductive step: suppose true for nxn matrices, show for $(n+1)(n+1)$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1,n+1}C_{i,n+1} = a_{11}C_{11}$$ (cofactors)

$C_{ik} = 0$ for all $k=1$

$= a_{11}a_{22}\cdots a_{n+1,n+1}$ □

for nxn:

SWAP ROWS: $\Rightarrow$ det mult. by (-1)

consequence: if A has repeated row, $\det|A| = 0$

$\det A = \det A^T$

$\det(AB) = \det(A)\cdot\det(B)$

consequence: if Q is orthogonal, $\det(Q) = \pm 1$

$Q^TQ = I \Rightarrow \det(Q^T)\cdot\det(Q) = 1$

Ex: $\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 5$ note: adding mult. of one row to another doesn't change det

gaussian elim:

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{vmatrix}$$

→ product of pivots: $2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4} = 5$

$$\begin{vmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{vmatrix} = \begin{vmatrix} \vec{r}_1 \\ 2\vec{r}_1 + \vec{r}_2 \\ \vec{r}_3 \end{vmatrix} = 2\begin{vmatrix} \vec{r}_1 \\ \vec{r}_1 \\ \vec{r}_3 \end{vmatrix} + \begin{vmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{vmatrix} = \begin{vmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{vmatrix}$$

*good trick, but don't
do computationally

CRAMER'S RULE:

solve $A\vec{x}=\vec{b}$ with dets

$A=(\vec{a}_1,\dots,\vec{a}_n)$, $B_1=(\vec{b}\ \vec{a}_2\ \dots\ \vec{a}_n)$, $M_1=(\vec{x}\ \vec{e}_2\ \dots\ \vec{e}_n)$

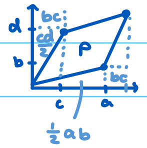
$AM_1=(A\vec{x}\ \ A\vec{e}_2\ \cdots\ A\vec{e}_n)=(\vec{b}\ \vec{a}_2\ \cdots\ \vec{a}_n)=B_1$ $\quad\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & & \vdots \\ \vdots & & 1 & 0 \\ x_n & 0 & \ddots & 1 \end{pmatrix}$

$\det(A)\cdot\det(M_1)=\det(B_1)$, $\det(M_1)=x_1$

$$\Rightarrow x_1=\frac{\det(B_1)}{\det(A)}$$

$x_k=\dfrac{\det(B_k)}{\det(A)}$, where B_k is A with kth col replaced by $\vec{b}$

AREA & DETERMINANT:



$\text{area}(P)=(b+d)(a+c)-2bc-ab-cd$

$=ad-bc$

* in 2x2 case, det says smthg ab. area

area in $\mathbb{R}^2=|\det(\vec{v}_1\ \vec{v}_2)|$

volume in $\mathbb{R}^3=|\det(\vec{v}_1\ \vec{v}_2\ \vec{v}_3)|=|\vec{v}_1\cdot(\vec{v}_2\times\vec{v}_3)|$

$\det(\vec{e}_1\ \dots\ \vec{e}_n)=1$
$\Leftrightarrow$ box w/ side len. 1 has volume 1

volume of parallelpiped made out of $\vec{v}_1,\dots,\vec{v}_n$ is $|\underbrace{\det(\vec{v}_1\ \vec{v}_2\dots\vec{v}_n)}_{\text{edge matrix}}|$

$|\det(a)|=1\Rightarrow$ not changing val (rotating it, etc)

10/20 **EIGENVALUES:**

scalar λ is eigenvalue of square matrix A if there is a

non-zero vector $\vec{v}$ s.t. $A\vec{v}=\lambda\vec{v}$ eigenvector

· some direction in which i apply a $\overset{A}{\text{matrix}}$, $\vec{v}$ doesn't change dirs. it just scales

note: some A have no real λ:

$A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ rotates by 90° → no direction

note: if $N(A)\neq\{\vec{0}\}$, $\lambda=0$ is eigenvalue of A

$\vec{v}\in N(A)\Rightarrow A\vec{v}=0=0\cdot\vec{v}$

Ex: I_n has $\lambda=1$ → $I_n\cdot\vec{v}=1\cdot\vec{v}$ for all $\vec{v}$

$\mathbb{R}^n$ is the eigenspace associated w/ $\lambda=1$

EIGENSPACE: span of eigenvectors associated w/ λ

$A\vec{v}=\lambda\vec{v}\Leftrightarrow(A-\lambda I_n)\vec{v}=\vec{0}$

$\det(A-\lambda I_n)=0\Leftrightarrow\lambda$ is eigenvalue

Ex: $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\Rightarrow A-\lambda I_n=\begin{pmatrix}-\lambda&1\\-1&-\lambda\end{pmatrix}$, $\det(A-\lambda I_n)=\lambda^2+1=0$ → no real solns. $\lambda=\pm i$, $i=\sqrt{-1}$

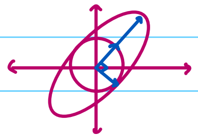
$\lambda=i$: $A-iI=\begin{pmatrix}-i&1\\-1&i\end{pmatrix}$, $\begin{pmatrix}i\\-1\end{pmatrix}$ in nullspace $A-I$

$\lambda=-i$: $A+iI=\begin{pmatrix}i&1\\-1&i\end{pmatrix}$, $\begin{pmatrix}1\\-i\end{pmatrix}$ in nullspace $A-I$

$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}i\\-1\end{pmatrix}=i\begin{pmatrix}i\\-1\end{pmatrix}$, $\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=-i\begin{pmatrix}1\\-i\end{pmatrix}$

can $A\in\mathbb{R}^{3\times3}$ have no real eigenvalues? → no! $\det(A-\lambda I)$ is deg 3, always has real root
→ 3D matrix always has a real λ. (all odds, not even).

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 \rightarrow \lambda - 2 = \pm 1 \rightarrow \lambda = 1,\ \lambda = 3$



eigenvectors: $\lambda=1$: $A-I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in $N(A-I)$ note: $\vec{v}_1 \perp \vec{v}_2$

$\lambda=3$: $A-3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in $N(A-I)$

$A^{100}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ → scalar is still just 1

$A^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A^{100}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = A^{100}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = 3^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

nxn matrix with n distinct eigenvalues diagonalizes

$\lambda_1, \dots, \lambda_n$ distinct eigenvalues

$\vec{v}_1, \dots, \vec{v}_n$ eigenvectors for each λ $AX = X\Lambda$

$X = (\vec{v}_1, \dots, \vec{v}_n)$ $A\vec{x} = (A\vec{v}_1, \dots, A\vec{v}_n) = (\lambda_1\vec{v}_1, \dots, \lambda_n\vec{v}_n)$

$= (\vec{v}_1, \dots, \vec{v}_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$= \vec{x}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

claim: if evals are distinct, X is invertible

X invertible, $AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$

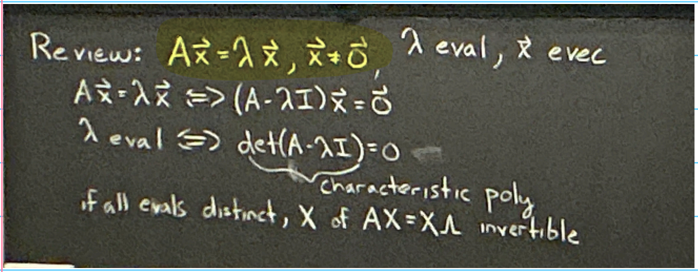
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, 0 is a λ, $\vec{v} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ eigenvector

Pf (2x2 case): suppose $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

$(A-\lambda_1 I_2)(c_1\vec{v}_1 + c_2\vec{v}_2) = c_2(A-\lambda_1 I_2)\vec{v}_2 = c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0} \Rightarrow c_2 = 0$

same arg. w/ $(A-\lambda_1 I_2) \Rightarrow c_1 = 0$

10/22



Review: $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$, λ eval, $\vec{x}$ evec

$A\vec{x} = \lambda\vec{x} \Leftrightarrow (A-\lambda I)\vec{x} = \vec{0}$

λ eval $\Leftrightarrow \underbrace{\det(A-\lambda I)}_{\text{characteristic poly}} = 0$

if all evals distinct, X of $AX = X\Lambda$ invertible

DIAGONALIZATION:

$AX = X\Lambda$, X invertible → $A = X\Lambda X^{-1}$

→ $A^m = X\Lambda^m X^{-1}$

A & $C = B^{-1}AB$ have same evals

$C = B^{-1}(X\Lambda X^{-1})B = (B^{-1}X)\Lambda(X^{-1}B)$ → A & C similar

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $A^m = ?$, A upper $\Delta \Rightarrow \lambda = 1$ & $\lambda = 3$

$\lambda=1$: $A - I\lambda = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda=3$: $A - I\lambda = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$A^m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^m \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3^m \\ 0 & 3^m \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3^m - 1 \\ 0 & 3^m \end{pmatrix}$

DETERMINANT & TRACE:

recall: $\det(AB) = \det(A)\det(B)$

$\det(A^{-1}) = \frac{1}{\det(A)}$ $\quad AA^{-1} = I$

$\det(A) \cdot \det(A^{-1}) = \det(I) = 1$

if $A = X\Lambda X^{-1}$, Λ diagonal

$\det(A) = \det(X\Lambda X^{-1}) = \det(X)\det(\Lambda)\det(X^{-1}) = \det(\Lambda) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

trace of $A \in \mathbb{R}^{n\times n}$ is $tr(A) = A_{11} + A_{22} + \dots + A_{nn}$ satisfies $tr(\underset{m\times n}{A}\underset{n\times m}{B}) = tr(\underset{n\times m}{B}\underset{m\times n}{A})$

$tr(A) = tr(X\Lambda X^{-1}) = tr(X^{-1}X\Lambda) = tr(\Lambda) = \lambda_1 + \lambda_2 + \dots + \lambda_n$

$tr(A^k) = tr(\Lambda^k) = \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k$

FIBONAUCCI #s:

$\underset{F_0}{0}, \underset{F_1}{1}, 1, 2, 3, 5, 8, \dots$ $\quad F_{k+2} = F_{k+1} + F_k$

$\vec{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$

$$\vec{u}_{k+1} = \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} F_{k+1} + F_k \\ F_{k+1} \end{pmatrix} = \overset{A}{\overset{\parallel}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}}\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = A\vec{u}_k, \quad \vec{u}_0 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\det(A - I\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \rightarrow \lambda = \frac{1+\sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \cdot \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \cdot \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \rightarrow X = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$$

EXAM REVIEW: II

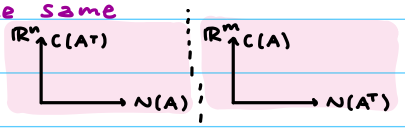
- vector space closed under $c_1\vec{v}_1 + c_2\vec{v}_2$
- basis: lin. ind. vectors spanning vect. space.
  - lin. ind. is ind. of idea of dot prod.
  - any vector space can have basis
- complete soln: used RREF & elim. can scale.
  - got special solns (basis to nullspace)
- projections: $P = A(A^TA)^{-1}A^T$ proj onto $C(A)$
  - orthonormal $Q \in \mathbb{R}^{n\times k}$ → $P = QQ^T$ (columns = orthonormal)
- gram schmidt: $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ ← need basis for space (need lin. ind)

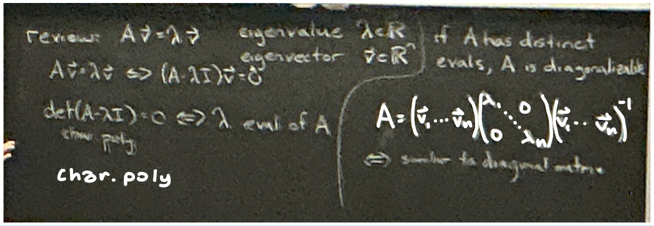
  $\vec{w}_1 = \vec{v}_1$ (if not $\vec{0}$)

  $\vec{w}_2 = \vec{v}_2 - \frac{\vec{w}_1\vec{w}_1^T}{\vec{w}_1^T\vec{w}_1}\vec{v}_2$

☆ • solve $A^TA\vec{x} = A^T\vec{b}$ ← least sqs

- $Q \in \mathbb{R}^{n\times n}$ → $Q^TQ = I$, $QQ^T = I$ ☆
  - length & angle kept the same
- FTLA: A $(m\times n)$ $\mathbb{R}^m \rightarrow \mathbb{R}^n$





review: $A\vec{v}=\lambda\vec{v}$ eigenvalue $\lambda\in\mathbb{R}$, eigenvector $\vec{v}\in\mathbb{R}^n$

$A\vec{v}=\lambda\vec{v} \Leftrightarrow (A-\lambda I)\vec{v}=\vec{0}$

$\det(A-\lambda I)=0 \Leftrightarrow \lambda$ eval of A

char. poly

char. poly

if A has distinct evals, A is diagonalizable

$A=(\vec{v}_1 \cdots \vec{v}_n)\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}(\vec{v}_1 \cdots \vec{v}_n)^{-1}$

$\Leftrightarrow$ similar to diagonal matrix

10/27

S symm. $\Rightarrow$ $\vec{v}$'s for diff. λ's are $\perp$

Lm: $S=S^T$, if $\lambda_1 \neq \lambda_2$, then $\vec{v}_1 \cdot \vec{v}_2 = 0$

Pf: $\vec{v}_1^T S \vec{v}_2 = \vec{v}_1^T(\lambda_2\vec{v}_2) = \lambda_2 \vec{v}_1^T\vec{v}_2$

$\vec{v}_1^T S \vec{v}_2 = (S^T\vec{v}_1)^T\vec{v}_2 = (S\vec{v}_1)^T\vec{v}_2 = (\lambda_1\vec{v}_1)^T\vec{v}_2 = \lambda_1\vec{v}_1^T\vec{v}_2$

$\Rightarrow (\lambda_1-\lambda_2)\vec{v}_1^T\vec{v}_2 = 0$ $\quad \lambda_1 \neq \lambda_2 \Rightarrow \vec{v}_1^T\vec{v}_2 = 0$

Ex: $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ → eigen pairs $\lambda=2, \begin{pmatrix}1\\1\end{pmatrix}$ $\quad 0, \begin{pmatrix}1\\-1\end{pmatrix}$

orthogonal

say a matrix A preserves subspace V if $\vec{v}\in V$, $A\vec{v}\in V$ for $A\in\mathbb{R}^{n\times n}$

$\{0\}$ is preserved

$V\in\mathbb{R}^n$ preserved

if $\vec{v}$ is e. vec. of A, span$\{\vec{v}\}$ preserved

Lm: $S=S^T$ and V is preserved, then $V^\perp$ is too.

Pf: let $\vec{v}\in V$ and $w\in V^\perp$

know $S\vec{v}\in V \Rightarrow \vec{w}^T S\vec{v}=0$

$\Rightarrow \vec{w}^T S^T \vec{v} = 0 \Rightarrow (S\vec{w})^T\vec{v}=0 \rightarrow S\vec{w}\in V^\perp$ ▫

Ex: $\begin{pmatrix}1&1\\1&1\end{pmatrix}$

$V=\left\{a\begin{pmatrix}1\\-1\end{pmatrix} a\in\mathbb{R}\right\} \rightarrow V^\perp = \left\{b\begin{pmatrix}1\\1\end{pmatrix} \middle| b\in\mathbb{R}\right\}$ also preserved

Ex: $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\quad V=\left\{a\begin{pmatrix}1\\0\end{pmatrix} \middle| a\in\mathbb{R}\right\}$ is preserved

$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ $\quad V^\perp=\left\{b\begin{pmatrix}0\\1\end{pmatrix} \middle| b\in\mathbb{R}\right\}$ is not! $\quad \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \notin V^\perp$

NOT sym, so property doesn't hold

QUADRATIC FORMS

for $S\in\mathbb{R}^{n\times n}$, $S=S^T$ define $f(x_1,\ldots,x_n)=\vec{x}^T S\vec{x}$

Ex: $S=\begin{pmatrix}a&b\\b&c\end{pmatrix}$, $f=(x \;\; y)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = (x \;\; y)\begin{pmatrix}ax+by\\bx+cy\end{pmatrix}$

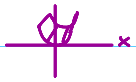
$f(x,y)=ax^2+2bxy+cy^2$ ← tells you ab. shape of curve (Hessian)

def: S is positive definite if $f(\vec{x})>0$ for $\vec{x}\neq 0$

Ex: $S=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ → $f=x^2+y^2$ (pos. def)

Ex: $S=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ not pos. def! $f=x^2-y^2$

Ex: $S=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ is positive semi-definite: f is always ≥ 0 → $f(x,y)=x^2$



EX: $f(x,y) = ax^2 + 2bxy + cy^2$

$\vec{\nabla} f = \begin{pmatrix} 2ax + 2by \\ 2bx + 2cy \end{pmatrix} = 2s \begin{pmatrix} x \\ y \end{pmatrix}$ → for general n, $\vec{\nabla} f = 2S\vec{x}$ ← symmetry

take partials

EX: $I_n \rightarrow g(\vec{x}) = x_1^2 + x_2^2 + \dots + x_n^2$

$\nabla g(\vec{x}) = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \end{pmatrix} = 2\vec{x} = 2 I_n \vec{x}$

CONSTRAINED OPTIMIZATION (λ-mult.)

max $f(\vec{x})$

$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 1 \end{cases}$ find λ, $\vec{x}$ satisfying this

Subject to $\overset{g(\vec{x})}{\overset{\parallel}{\|\vec{x}^2\|}} = 1$

EX: $\vec{\nabla} f = 2S\vec{x}$, $\vec{\nabla} g = 2\vec{x} \rightarrow 2S\vec{x} = 2\lambda\vec{x} \rightarrow S\vec{x} = \lambda\vec{x}$

at max of $f(\vec{x})$ on $\|\vec{x}\|^2 = 1$, we have an eigen vect $\vec{x}$

we just showed: $S = S^T$ means S has @ least one eval λ and evec $\vec{x}$

10/29 last time: $S^T = S \Rightarrow S$ has @ least one e. val & e. vec $\vec{v}$

EXAM 2: #Q2b, 3b, 4b

$S^T = S$, S preserves $V \Rightarrow S$ preserves $V^\perp$

EXAM 2 POST-MORTEM:

$A\vec{x} = \vec{b}$, $\vec{x}_p + \vec{x}_n$, $\vec{x}_n \in N(A)$ $A\vec{x}_p = \vec{b}$

↳ set $= 0$

$A\vec{x} \in C(A)$

SPECTRAL THM: $S = S^T \Rightarrow$ there is an orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ of eigenvectors

$Q = (\vec{v}_1, \dots, \vec{v}_n)$ $SQ = Q\Lambda$, $Q^{-1} = Q^T$ $\Rightarrow S = Q\Lambda Q^{-1}$

PF (by induction): $n=1$ $S = (1)(S)(1)^T$ ✓

IND STEP: assume true for n, show for $n+1$ $S \in \mathbb{R}^{(n+1)(n+1)}$, $S \in S^T$

quad. form: get eigenpair λ_{n+1}, $\vec{v}_{n+1}$ for S, $\|\vec{v}_{n+1}\| = 1$ $\Rightarrow S$ preserves span$\{\vec{v}_{n+1}\}^\perp$

Gram-Schmidt $\Rightarrow \vec{y}_1, \dots \vec{y}_n$ on basis for span$\{\vec{v}_{n+1}\}^\perp$

$\Rightarrow \beta = \{\vec{y}_1, \dots, \vec{y}_n, \vec{v}_{n+1}\}$ is on basis $\mathbb{R}^{n+1}$

what does S look like in this basis?

Ex: $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \vec{y}_1 + \vec{v}_{n+1}$

i.e. $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_\beta = x_1\vec{y}_1 + x_2\vec{y}_2 + \dots + x_n\vec{y}_n + x_{n+1}\vec{v}_{n+1}$

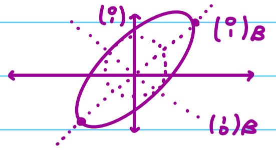
$S\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}_\beta = S\vec{v}_{n+1} = \underset{\overset{\uparrow}{\text{e. val}}}{\lambda_{n+1}} \vec{v}_{n+1} = \lambda_{n+1}\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}_\beta$

S in this basis looks like $S = \left(\begin{array}{c|c} \begin{matrix} S \text{ on} \\ \text{span}\{\vec{v}_{n+1}\}^\perp \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots\cdots 0 & \lambda_{n+1} \end{array}\right)_\beta$

by induction, true for S restricted to span$\{\vec{v}_{n+1}\}$ □

EX: $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ have eg pairs $1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{Q^T}$ in basis $\mathcal{B} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}_{\mathcal{B}}$



$S^k \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} x \\ 3^k y \end{pmatrix}_{\mathcal{B}}$

* long-term dynamics

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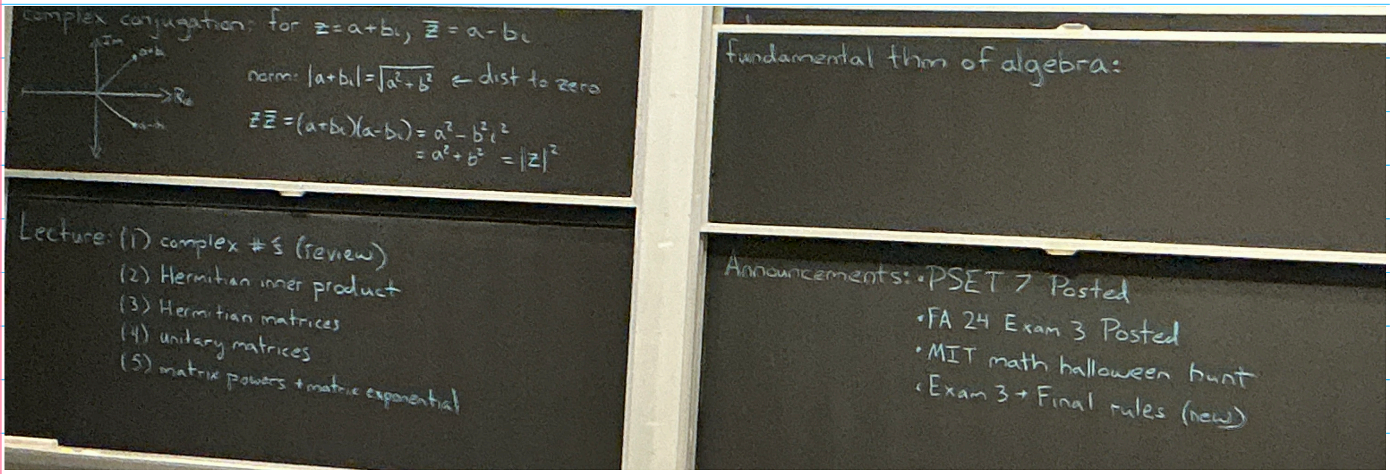


complex conjugation: for $z = a+bi$, $\bar{z} = a - bi$

norm: $|a+bi| = \sqrt{a^2+b^2}$ ← dist to zero

$z\bar{z} = (a+bi)(a-bi) = a^2 - b^2 i^2$
$= a^2 + b^2 = |z|^2$

fundamental thm of algebra:

Lecture: (1) complex #s (review)
(2) Hermitian inner product
(3) Hermitian matrices
(4) unitary matrices
(5) matrix powers + matrix exponential

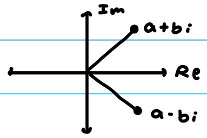
Announcements:
- PSET 7 Posted
- FA 24 Exam 3 Posted
- MIT math halloween hunt
- Exam 3 → Final rules (new)

complex #s:

$z = a + bi$ satisfies $i^2 = -1$ $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$

$+, -, \times$ same $a, b \in \mathbb{R}$

complex conjugation: for $z = a+bi$, $\bar{z} = a - bi$



norm: $|a+bi| = \sqrt{a^2+b^2}$ ← dist to 0

$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 = |z|^2$

division: $\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z\bar{w}}{\|w\|^2}$

useful facts: $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$

euler formula: for $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow |e^{i\theta}| = 1$

FUNDAMENTAL THM OF ALGEBRA:

every complex polynomial of degree n has n complex roots (possibly w/ repetition)

HERMITIAN INNER PRODUCT

if $\vec{v}, \vec{w} \in \mathbb{C}^n$, $\vec{v} \cdot \vec{w} = \overline{\vec{w}^T} \vec{v} = \bar{w}_1 v_1 + \bar{w}_2 v_2 + \ldots + \bar{w}_n v_n \in \mathbb{C}$

define $\vec{v}^* = \overline{\vec{v}^T}$, $\vec{v}, \vec{w}$ orthogonal if $\vec{w}^* \vec{v} = 0$

$\vec{v}^* \vec{v} = |v_1|^2 + \ldots + |v_n|^2 = \|v_n\|^2 = \|\vec{v}\|^2 \in \mathbb{R}$

note: $\vec{v}^* \vec{w} \neq \vec{w}^* \vec{v}$, but they are complex conjugates. $\vec{v}^* \vec{w} = \overline{\vec{w}^* \vec{v}}$

for $A \in \mathbb{C}^{m \times n}$, $A^* = \overline{A^T}$ (sometimes written A^H)

HERMITIAN MATRIX:

$A \in \mathbb{C}^{n \times n}$ is Hermitian if $A^* = A$ (complex analogue of real symmetric)

on diagonal, $A_{ii} = \overline{A_{ii}} \Rightarrow A_{ii} \in \mathbb{R}$

Ex: $A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$ Hermitian

e. vals: $\det(A - \lambda I) = 0 \rightarrow \begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = 10 - 7\lambda + \lambda^2 - (9+9) = \lambda^2 - 7\lambda - 8 = 0 \rightarrow (\lambda - 8)(\lambda + 1) \rightarrow \lambda = -1, 8$

$\lambda = -1$: $A - (-1)I = A + I = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$

$\lambda = 8$: $A - (8)I = A - 8I = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix} \rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

2 linearly ind. eigenvectors

note: $\vec{v}_1^* \vec{v}_2 = (1+i \quad -1)\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = 0$

for Hermitian matrix A, (1) e. vals are real
(2) e. vecs for distinct evals are ⊥

UNITARY MATRIX:

$U \in \mathbb{C}$ is unitary if $U^* U = I_n \Leftrightarrow U^{-1} = U^*$ *every orthog. matrix is unitary

unitary matrices preserve length & angle

Ex: $n = 1$ $U = e^{i\theta}$, $U^* U = I$

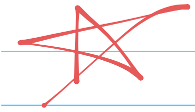
$e^{i\theta}(e^{-i\theta}) = 1$

have a whole 2π choice of angles in 1D complex

1D: $Q = [\pm 1]$

$\|U\vec{x}\|^2 = (U\vec{x}) \cdot (U\vec{x}) = \vec{x}^* U^* U \vec{x} = \vec{x}^* \vec{x} = \|\vec{x}\|^2$

$(U\vec{x}) \cdot (U\vec{y}) = \vec{y}^* U^* U \vec{x} = \vec{y}^* \vec{x} = \vec{x} \cdot \vec{y}$



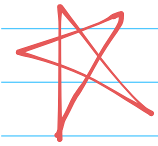
Ex: $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $U = \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$ discrete Fourier matrix. $U_{jk} = w^{(j-1)(k-1)}$ $w = e^{2\pi i/n}$

each col = 1, i, −1, −i to the 0, 1, etc.
(depending on col.)

11/3 MATRIX POWERS + EXPONENTIAL:

recap: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots$

$d/dx\,(e^{\lambda x}) = \lambda e^{\lambda x}$



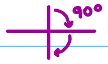
Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, etc.

$e^{tA} := I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \ldots$, t scalar

for this A, $e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $e^{-tA} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$

$e^{tA}e^{-tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 

$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \quad A^3 = A^2 A = -A, \quad A^4 = I$

$A^5 = A, \quad A^6 = -I, \quad A^7 = -A, \quad A^8 = I, \ \ldots$

$$e^{At} = I + tA + \frac{t^2}{2}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!} I + \frac{t^5}{5!} A + \frac{t^6}{6!}(-I) + \frac{t^7}{7!}(-A) + \frac{t^8}{8!} I + \ldots$$

★ $$= I\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \ldots\right) + A\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right)$$

$e^{tA} = I\cos t + A\sin t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ ← rotation matrix by t

PROPERTIES: (1) $\frac{d}{dt} e^{tA} = A e^{tA}$

$$\frac{d}{dt} e^{tA} = \frac{d}{dt}\left(I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \ldots\right) = \left(A + tA^2 + \frac{t^2}{2} A^3 + \ldots\right)$$

(2) $(e^{tA})^{-1} = e^{-tA}$

(3) if $\overbrace{A^T = -A}^{\text{skew-symmetric}} \rightarrow e^{tA}$ is orthogonal

(4) $A\vec{x} = \lambda \vec{x} \Rightarrow e^{tA}\vec{x} = e^{t\lambda}\vec{x}$

$$A\vec{x} = \lambda\vec{x} \Rightarrow e^{tA}\vec{x} = I\vec{x} + tA\vec{x} + \frac{t^2}{2}A^2\vec{x} + \frac{t^3}{3!}A^3\vec{x} + \ldots$$

$$= \vec{x} + (t\lambda)\vec{x} + \frac{t^2}{2}\lambda^2\vec{x} + \frac{t^3}{3!}\lambda^3\vec{x} + \ldots$$ ← all these terms are scaled versions of x

$$= \left(1 + (t\lambda) + \frac{(t\lambda)^2}{2} + \frac{(t\lambda)^3}{3!} + \ldots\right)\vec{x} = e^{t\lambda}\vec{x}$$

★ DIFFERENTIAL EQUATIONS:

$f''(t) = af(t) + bf'(t) \quad a, b \in \mathbb{R}$

$\vec{u}(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} \quad \vec{u}'(t) = \begin{pmatrix} f'(t) \\ f''(t) \end{pmatrix} = \overset{A}{\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}} \overset{\vec{u}(t)}{\begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}}$ $\quad \vec{u}' = A\vec{u} \ \curvearrowright$ Soln: $\vec{u}(t) = e^{tA}\vec{u}(0)$

$\vec{u}'(t) = Ae^{tA}\vec{u}(0)$

$f''' = af + bf' + cf''$

$\vec{u} = \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} \quad \vec{u}' = \begin{pmatrix} f' \\ f'' \\ f''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix} \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}$ Soln $\vec{u}(t) = e^{tA}\vec{u}(0)$ $\quad \begin{pmatrix} 0 & 1 & \ddots & 0 \\ a_1 & \cdots & & a_n \end{pmatrix}$

Ex: $f(t)$ solves $\begin{cases} f''(t) = -f(t) \\ f(0), f'(0) \text{ given} \end{cases}$

$\vec{u} = \begin{pmatrix} f \\ f' \end{pmatrix} \quad \vec{u}' = \begin{pmatrix} f' \\ f'' \end{pmatrix} = \underset{A}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} f \\ f' \end{pmatrix}$

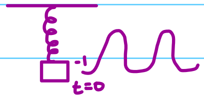
soln: $\vec{u}(t) = e^{tA}\vec{u}(0)$

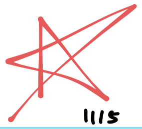
$\vec{u}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$ ★ good test question

↑computed earlier

$f(t) = (\cos t) f(0) + (\sin t) f'(0)$

spring oscillation





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SINGLE VALUE DECOMPOSITION:

$A = U \Sigma V^T$, $A \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ ← U & V both orthogonal

$m>n$: $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \sigma_n \\ \vdots & 0 & \end{pmatrix}$, $m<n$: $\Sigma = \begin{pmatrix} \sigma_1 & \cdots & 0 & \vdots & 0 \\ 0 & \cdots & \sigma_n & \vdots & \end{pmatrix}$ orthogonal, $\Sigma \in \mathbb{R}^{m\times n}$ "diagonal" ($\Sigma_{ij}=0$ for $i \neq j$)

Ex $A = \begin{pmatrix} 5 & 4 \\ 0 & 3 \end{pmatrix}$ goal: find $\vec{v}_1, \vec{v}_2$ orthonormal s.t. $(A\vec{v}_1)\cdot(A\vec{v}_2)=0$

set $\sigma_1 = \|A\vec{v}_1\|$, $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$

$\sigma_2 = \|A\vec{v}_2\|$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$

"good guess": $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to A\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 5 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 9 \\ 3 \end{pmatrix}}^{\text{these are } \perp \text{!}} \Rightarrow \sigma_1^2 = \|A\vec{v}_1\|^2 = \frac{81}{2} + \frac{9}{2} = 45 \to \sigma_1 = \sqrt{45} = 3\sqrt{5}$

$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to A\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 5 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \Rightarrow \sigma_2^2 = \|A\vec{v}_2\|^2 = \frac{1}{2} + \frac{9}{2} = 5 \to \sigma_2 = \sqrt{5}$

$\vec{u}_1 = \frac{1}{\sqrt{2}\cdot\sqrt{45}}\begin{pmatrix} 9 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

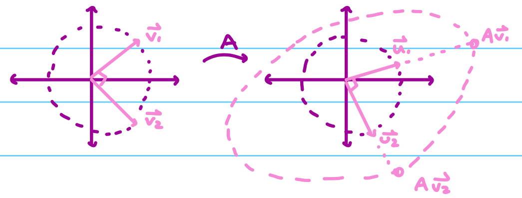
$U = (\vec{u}_1\ \vec{u}_2)$, $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $V = (\vec{v}_1\ \vec{v}_2)$

$A = U\Sigma V^T = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix}\begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & 3 \end{pmatrix}$

*change of basis from input & output

orthog elipsis

input:



$(\vec{u}_1\ \vec{u}_2)\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} = (\sigma_1\vec{u}_1\quad \sigma_2\vec{u}_2)\begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} = \overbrace{\sigma_1\vec{u}_1\vec{v}_1^T} + \sigma_2\vec{u}_2\vec{v}_2^T$

$\sigma_1\vec{u}_1\vec{v}_1^T = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$, $\sigma_2\vec{u}_2\vec{v}_2^T = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$

for A, $\text{rank}(A) = r$, $A = \sigma_1\vec{u}_1\vec{v}_1^T + \ldots + \sigma_r\vec{u}_r\vec{v}_r^T$ where $\{\vec{u}_i\}_{i=1}^r$, $\{\vec{v}_i\}_{i=r}^r$ are on basis

and $A\vec{v}_i = \sigma_i\vec{u}_i$, $i = 1, \ldots, n$, $\sigma_i = 0$ for $i > r$

σ_i = singular values

$\vec{u}_i$ = left singular vectors, $\vec{v}_i$ right singular vectors

$A = U\Sigma V^T$, $C(U) = \mathbb{R}^m$, $C(V) = \mathbb{R}^m$

$\Rightarrow \text{rank}(A) = \text{rank}(\Sigma)$

$\Rightarrow \Sigma$ has exactly r non-zero diagonal elements

HOW TO FIND SVD:

if $A = U\Sigma V^T$, $A^TA = (U\Sigma U^T)^T U\Sigma V^T = V\Sigma^T \overbrace{U^T U}^{I_m} \Sigma V^T = V(\Sigma^T\Sigma)V^T$

$\Sigma^T\Sigma \in \mathbb{R}^{n\times n}$ diagonal

proof → $(\Sigma^T\Sigma)_{ij} = \sum_{k=i}^{m} (\Sigma^T)_{kj} = (\Sigma^T)_{ij}\,\Sigma_{jj}$ only when $i=j$

$(A^TA)^T = A^TA$

SPECTRAL THM: orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ of eigenvectors with eigenvalues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

A^TA is positive semi-definite $(A\vec{v})^TA\vec{v}$

$A^TA\vec{v} = \lambda\vec{v} \Rightarrow \vec{v}^TA^TA\vec{v} = \lambda\vec{v}^T\vec{v} = \|A\vec{v}\|^2 = \lambda\|\vec{v}\|^2 \rightarrow \lambda = \frac{\|A\vec{v}\|^2}{\|\vec{v}\|^2} \geq 0$ $\lambda = \sigma^2$

define $\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$ ⋮ $\vec{u}_i^T\vec{u}_j = \frac{(A\vec{v}_i)^TA\vec{v}_j}{\sigma_i\ \sigma_j} = \frac{\vec{v}_i^T(A^TA)\vec{v}_j}{\sigma_i\ \sigma_j}$

$= \vec{v}_i^T\vec{v}_j\frac{\sigma_j^2}{\sigma_i\sigma_j} = 0$ ← u_i's are all ⊥

Ex: $A = \begin{pmatrix} 5 & 4 \\ 0 & 3 \end{pmatrix} \rightarrow A^TA = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{vmatrix} \rightarrow (25-\lambda)^2 = 20^2 \rightarrow \lambda = 5, 45$

$\lambda = 45$: $A^TA - 45I = \begin{pmatrix} -20 & 20 \\ 20 & -20 \end{pmatrix} \rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 5$: $A^TA - 5I = \begin{pmatrix} 20 & 20 \\ 20 & 20 \end{pmatrix} \rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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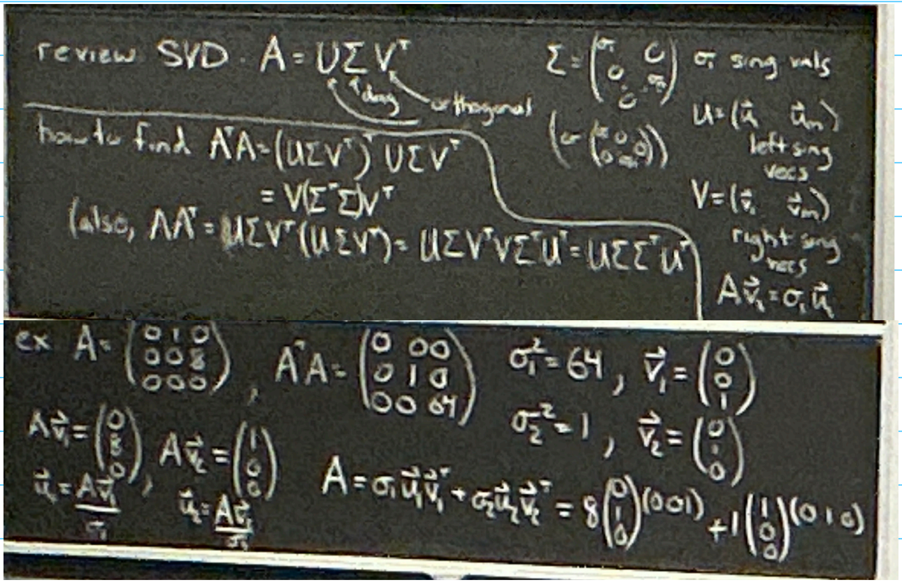


review SVD $A = U\Sigma V^T$ $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \ddots & \sigma_n \end{pmatrix}$ σ_i sing vals

diag orthogonal

how to find $A^TA = (U\Sigma V^T)^T U\Sigma V^T$ $\left(\text{or} \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \sigma_n & 0 \end{pmatrix}\right)$ $U = (\vec{u}_1 \ \ \vec{u}_m)$ left sing vecs

$= V(\Sigma^T\Sigma)V^T$ $V = (\vec{v}_1 \ \ \vec{v}_m)$

(also, $AA^T = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma V^TV\Sigma^TU^T = U\Sigma\Sigma^TU^T$ right sing vecs

$A\vec{v}_1 = \sigma_1\vec{u}_1$

ex $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$, $A^TA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}$ $\sigma_1^2 = 64$, $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\sigma_2^2 = 1$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A\vec{v}_1 = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}$, $A\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2}$

$A = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T = 8\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\,0\,1) + 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(0\,1\,0)$

Ex: A^TA vs. AA^T

$A^TA\vec{v} = \lambda\vec{v} \Rightarrow AA^T(A\vec{v}) = \lambda(A\vec{v})$

if $A\vec{v}_i \neq 0$, $\lambda_i, A\vec{v}_i$ eigenpairs of AA^T * AA^T & A^TA have some non-zero e.vals

if $\lambda \neq 0 \Rightarrow \lambda\vec{v} = \vec{0} \Rightarrow A^T(A\vec{v}) \neq \vec{0} \rightarrow A\vec{v} \neq 0$

Ex: $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & -2 \end{pmatrix}$ $A^TA = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ $\lambda_1 = 8$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\lambda_3 = 0$

$AA^T = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ $\lambda_1 = 8$ $\frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda_2 = 2$, $e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{e}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$ $\frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\sigma_1 = \sqrt{8}$, $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $A = \sqrt{8}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}(0\ 0\ 1) + \sqrt{2}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}(1\ 0\ 0)$

$\sigma_2 = \sqrt{2}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{8} & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

PRINCIPLE COMPONENT ANALYSIS:

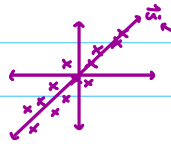
$A \in \mathbb{R}^{m\times n}$ is n data points $\text{col}_i(A) \in \mathbb{R}^m$,

with $AI = \sum_{i=1}^{n} \text{col}_i(A) = \vec{0}$

left singular vectors give principal components of data

Ex: $A = \mathbb{R}^{2\times n}$ $\overbrace{AA^T}^{\text{covariance matrix}} = \mathbb{R}^{2\times 2}$

↑ e.g. decomp.



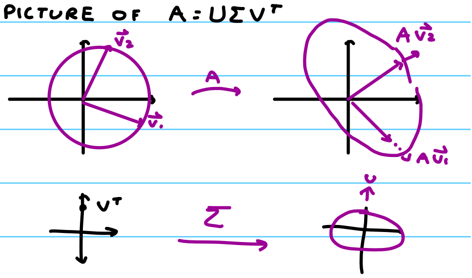
← telling us direction

11/12 **REVIEW:** **SVD:** $A = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T} = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_r \vec{u}_r \vec{v}_r^T$, $r = \text{rank}(A)$

how to find: $A^TA = \underset{\text{right sing. vecs}}{V}(\underset{\text{squared sing. vals}}{\Sigma^T\Sigma})V^T$ $\quad \vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$, $i = 1, \dots, r$

(or $Av = U\Sigma V^T V = U\Sigma$)

PICTURE OF $A = U\Sigma V^T$



NORM OF A MATRIX

for $A \in \mathbb{R}^{m\times n}$, $\text{rank}(A) = r$

a) **Frobenius norm:** $\|A\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$ ← what we've been working with whole time

b) **spectral norm:** $\|A\|_2 = \sigma_1 = \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_2}{\|x\|_2}$

c) **nuclear norm:** $\|A\|_N = \sigma_1 + \sigma_2 + \dots + \sigma_r$

note: $\|I_n\|_F = \sqrt{1 + \dots + 1} = \sqrt{n}$

$\|I_n\|_2 = 1$

$\|I_n\|_N = n$

$\|A\|_F$ also comes from dot prod: $A \cdot B = \text{trace}(A^TB) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$

$\|A\|^2 = A \cdot A = \text{trace}(A^TA) = \sigma_1^2 + \dots + \sigma_r^2$

$\|A\|_F^2 = \text{tr}(A^TA) = \sum_{j=1}^{n} \text{col}_j(A) \cdot \text{col}_j(A) = \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2$

n×n matrices is vector space

symmetric matrices $A = A^T$ subspace

closest symm. matrix is $(A + A^T)/2$

$$A = \underset{\text{symm.}}{\frac{A + A^T}{2}} + \underset{\text{skew symm.}}{\frac{A - A^T}{2}}$$

?
☆

Eckart - Young Theorem:

if B has $\text{rank}(B) = k$, then $\|A - B\|_F \geq \|A - (\sigma_1 \vec{u}_1 \vec{v}_1^T + \ldots + \sigma_k \vec{u}_k \vec{v}_k^T)\|_F$

* single best choice is the first k terms?

↑ exam 3 content

LINEAR TRANSFORMATIONS:

V, W vector spaces

linear transformation is map $T: V \to W$ with

a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all $\vec{v}, \vec{w} \in V$

b) $T(\alpha \vec{v}) = \alpha T(\vec{v})$ for all $\vec{v} \in V$, scalars α * if $\alpha = 0 \to$ linear trans. maps $\vec{0}$ to $\vec{0}$ $T(\vec{0}) = \vec{0}$
↳ $T(\vec{0}) = 0$!

Ex: $V = W = \mathbb{R}$ $T(x) = 2x$ ✓ yes linear transformation $T(x+y) = 2(x+y) = 2x + 2y = T(x) + T(y)$
$T(\alpha x) = 2\alpha x = \alpha T(x)$

Ex: $T(x) = x^2$ ✗ NO $T(2) = 4$, but $2T(1) = 2$ ✗

Ex: $T(x) = x + 1$ ✗ NO! $T(0) = 1$ ∴ can't satisfy $T(\vec{0}) = 0$

Ex: $T(x) = |x|$ ✗ NO! $T(0) = 0$, but $T(1) + T(-1) = 2$ ← should be 0 tho by rule of linearity

can represent $T: \mathbb{R}^n \to \mathbb{R}^m$ w/ matrix

$A = (T(\vec{e}_1)\ T(\vec{e}_2) \ldots T(\vec{e}_n)) \in \mathbb{R}^{m \times n}$

$T(\vec{x}) = T(x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n) = x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n) = A\vec{x}$

KERNEL: $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \in V$

IMAGE: $\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\} \subset W$

Ex: $V = \{$poly of deg ≤ 2 on $\mathbb{R}\}$, $W \in \mathbb{R}$

$T(p(x)) = p(1)$ ✓ yes L.T. $T(2 - x + 3x^2) = 4$

$\text{range}(T) = \mathbb{R}$ ← dim 1 possible basis: $\{1\}$

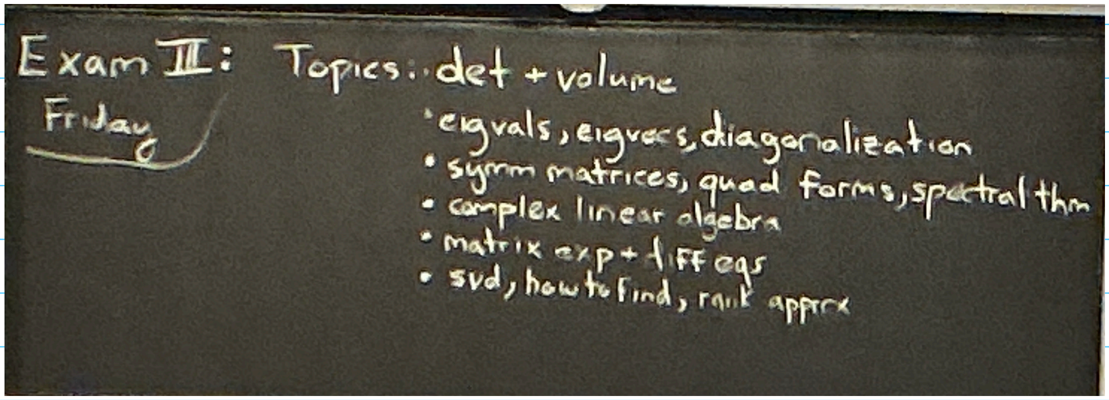
$\dim(V) = 3$ possible basis: $\{1, x, x^2\}$

FTLA "suggests" $\dim \ker(T) + \dim \text{range}(T) = \dim(V)$

T as a matrix in matrix in basis $\{1, x, x^2\}$

$T(a_1 + a_2 x + a_3 x^2) = a_1 + a_2 + a_3$ $A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a_1 + a_2 + a_3)$ $\to A = (1\ \ 1\ \ 1)$ $\dim(N(A)) = 2$
$\dim(C(A)) = 1$

11/14



Exam II: Friday

Topics: det + volume
- eigvals, eigvecs, diagonalization
- symm matrices, quad forms, spectral thm
- complex linear algebra
- matrix exp + diff eqs
- svd, how to find, rank apprx

Ex: V= { polynomials on $\mathbb{R}$ }

$p(x) \to p'(x)$ derivative is a linear map

using basis $B = \{1, x, x^2 \dots\}$

can encode derivative by matrix A with $\underline{A_{i,i+1}=i}$ and zero otherwise

$d/dx\,(2 - x + 3x^2) = -1 + 6x$

✪
$$\begin{array}{cc} & \begin{array}{ccc} 1 & x & x^2 \end{array} \quad \text{coeff.} \\ \begin{array}{c} i=1 \quad 1 \\ i=2 \quad x \\ i=3 \quad x^2 \end{array} & \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix} \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

CHANGE OF BASIS:

$T: \mathbb{R}^2 \to \mathbb{R}^2$ identity $T(\vec{x}) = \vec{x}$

in standard basis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

bases: input $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

output $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

T as a matrix in these bases:

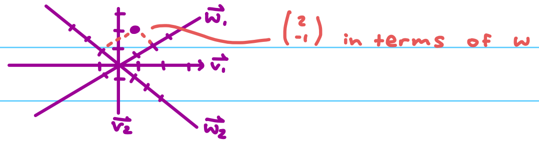
$T(\vec{v}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/2\,\vec{w}_1 + 1/2\,\vec{w}_2$

$T(\vec{v}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1/2\,\vec{w}_1 - 1/2\,\vec{w}_2$

$$\Rightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

Ex: if i want $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



in general, for $T: \mathbb{R}^n \to \mathbb{R}^m$ with $A \in \mathbb{R}^{m \times n}$, in std. basis

$\vec{v}_1, \dots, \vec{v}_n$ new input basis $\qquad V = (\vec{v}_1, \dots, \vec{v}_n) \in \mathbb{R}^{n \times n}$

$\vec{w}_1, \dots, \vec{w}_n$ new output basis $\qquad W = (\vec{w}_1, \dots, \vec{w}_n) \in \mathbb{R}^{m \times m}$

$\vec{v}_i = V\vec{e}_i$, $\vec{w}_j = W\vec{e}_j$

CLAIM: new matrix for T is $M = W^{-1}AV$

PF: want formula for M satisfying $T(\vec{v}_k) = \sum_{\ell=1}^{m} M_{\ell k}\vec{w}_\ell = W\text{col}_k(M) = \text{col}_k(WM)$

also, $T(\vec{v}_k) = A\vec{v}_k = A\,\text{col}_k(V) = \text{col}_k(AV)$

$\Rightarrow \text{col}_k(WM) = \text{col}_k(AV)$ for all $k \Rightarrow WM = AV$, $M = W^{-1}AV$ □

Ex: $A \in \mathbb{R}^{m \times n}$, $A = U\Sigma V^T$

new bases $\mathcal{U}$ output, V input

$M = U^TU\Sigma V^TV = \Sigma$

Ex: $A \in \mathbb{R}^{n \times n}$, $A = A^T$ $A = U\Lambda U^T$

new basis U (input + output)

$M = U^TU\Lambda U^TU = \Lambda$

for $A \neq A^T$, exists unitary matrix $\mathcal{U} \in \mathbb{C}^{n \times n}$ with $A = UTU^*$, where $T \in \mathbb{C}^{n \times n}$ is upper Δ (Schur form)

$\det(A - \lambda I) = \det(U(T - \lambda I)U^*)$ (Schur form)

$= \det(T - \lambda I) = \prod_{i=1}^{n}(T_{ii} - \lambda)$

if A has n distinct λ, $A = S\Lambda S^{-1}$ $S, \Lambda \in \mathbb{C}^{n \times n}$

new basis $M = S^{-1}S\Lambda S^{-1}S = \Lambda$ ← diagonal, but S is NOT $\perp$ (distorts S)

JORDAN FORM: what if A has repeated λs? Ex: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

how close can we get to diagonal?

$A = SJS^{-1}$, Jordan Form $\begin{pmatrix} J_1 & \cdots & 0 \\ 0 & \ddots & J_K \end{pmatrix}$ where $J \Delta \lambda$, or $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \ddots \end{pmatrix}$

Ex: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ — the 1 prohibits diag. matrix

Ex: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

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Ex: $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ not diagonalizable!

$S\begin{pmatrix} a & \\ 0 & a \end{pmatrix}S^{-1} = S(aI)S^{-1} = aI$

GOOD BASES FOR FUNCTION SPACES:

- polynomials
- trig poly (poly in cos/sin or in $e^{i\theta}$)

ORTHOGONALITY: $\int f(x)\overline{g(x)}\,dx = 0$ L^2 inner product

Ex: $e^{ik\theta}$ for $k=0,1,2\ldots$ on $[0,2\pi]$

$$\int_0^{2\pi} e^{ik\theta}\,\overline{e^{i\ell\theta}}\,d\theta = \int_0^{2\pi} e^{ik\theta}e^{-i\ell\theta}\,d\theta = \int_0^{2\pi} e^{i(k-\ell)\theta}\,d\theta = \begin{cases} \frac{1}{i(k-\ell)}e^{i(k-\ell)\theta}\Big|_0^{2\pi} = 0 & k\neq\ell \\ 2\pi & k=\ell \end{cases}$$

DERIVATIVE IN BASIS

f	f'
1	0
$e^{i\theta}$	$ie^{i\theta}$
$e^{2i\theta}$	$2ie^{2i\theta}$

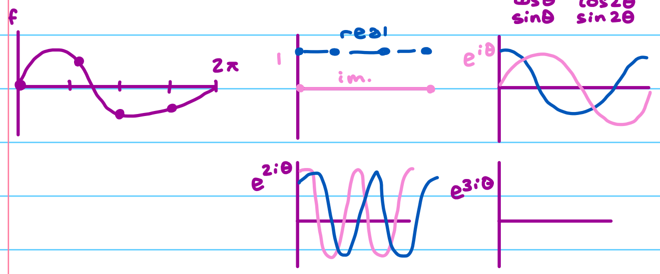
matrix

$$\begin{array}{c|ccccc} & 1 & e^{i\theta} & e^{2i\theta} & e^{3i\theta} & \ldots \\ \hline 1 & 0 & & & & \\ e^{i\theta} & & i & & 0 & \\ e^{2i\theta} & & & 2i & & \\ e^{3i\theta} & & 0 & & 3i & \\ \vdots & & & & & \ddots \end{array}$$

matrix for f'': $\begin{pmatrix} 0 & & & & \\ & -1 & & 0 & \\ & & -4 & & \\ & 0 & & -9 & \\ & & & & \ldots \end{pmatrix}$

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{array}{l} \leftarrow (1\ 1\ 1\ 1) \\ \leftarrow (e^{0i}\ e^{\pi/2i}\ e^{\pi i}\ e^{3/2\pi i}) \\ \leftarrow (e^{0 2i}\ e^{\pi/2/2i}\ e^{\pi/2i}\ e^{3\pi/2/2i}) \\ \\ \end{array} \\ 0\quad \pi/2\quad \pi\quad 3\pi/2 \end{array}$$

represent $\overset{\vec{x}}{\begin{pmatrix} f(0) \\ f(\pi/2) \\ f(\pi) \\ f(3\pi/2) \end{pmatrix}}$ in terms of $1, \underset{\substack{\cos\theta \\ \sin\theta}}{\overset{\uparrow}{e^{i\theta}}}, \underset{\substack{\cos 2\theta \\ \sin 2\theta}}{\overset{\uparrow}{e^{2i\theta}}}, \underset{\substack{\cos 3\theta \\ \sin 3\theta}}{\overset{\uparrow}{e^{3i\theta}}}$



$\vec{x} \rightarrow \begin{pmatrix} c_{\{1\}} \\ c_{\{e^{i\theta}\}} \\ c_{\{e^{2i\theta}\}} \\ c_{\{e^{3i\theta}\}} \end{pmatrix}$

Ex: Legendre poly: $P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2-1)$

satisfy $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$n=1$: $2P_2(x) = 3xP_1(x) - 1P_0(x) = 3x^2-1 \Rightarrow P_2(x) = \frac{1}{2}(3x^2-1)$

orthogonal on $[-1,1]$ scaled on $P_n(1) = 1$

$\int_{-1}^{1} x\,dx = 0$, $\int_{-1}^{1} \frac{1}{2}(3x^2-1)dx = \left(\frac{1}{2}x^3 - \frac{1}{2}x\right)\Big|_{-1}^{1} = 0$

Q: what poly of deg ≤ 2 minimizes $\int_{-1}^{1} |p(x)-e^x|^2\,dx$

use orthogonal basis! $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$

1: $\dfrac{\int_{-1}^{1} e^x dx}{\int_{-1}^{1} 1\cdot 1\,dx} = \dfrac{e-e^{-1}}{2}$

x: $\dfrac{\int_{-1}^{1} xe^x dx}{\int_{-1}^{1} x\cdot x\,dx} = 3e^{-1}x$

$\frac{1}{2}(3x^2-1)$: $\dfrac{\int_{-1}^{1}\frac{1}{2}(3x^2-1)dx}{\int_{-1}^{1}\left(\frac{1}{2}(3x^2-1)\right)^2dx}\,\frac{1}{2}(3x^2-1) = \frac{5}{2}(e-7e^{-1})\frac{1}{2}(3x^2-1)$

1st elts in taylor series:
$e^x = 1 + x + \frac{x^2}{2}$

best approx: $p(x) = \dfrac{e+e^{-1}}{2} + 3e^{-1}x + \dfrac{5(e-7e^{-1})(3x^2-1)}{4}$ * very strong approx!

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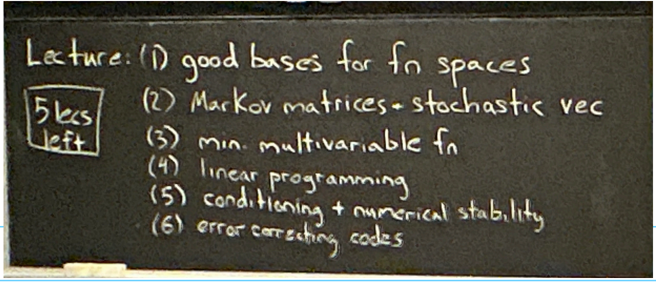


Lecture: (1) good bases for fn spaces

$\boxed{\text{5 lecs left}}$

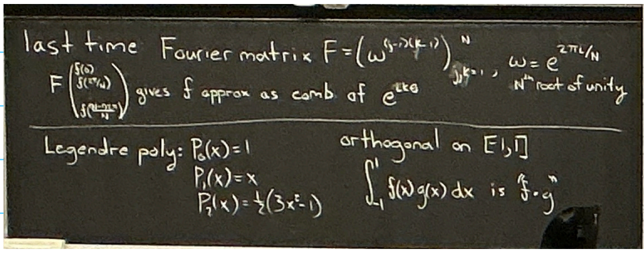
(2) Markov matrices + stochastic vec

(3) min. multivariable fn

(4) linear programming

(5) conditioning + numerical stability

(6) error correcting codes



last time Fourier matrix $F = \left(\omega^{(j-1)(k-1)}\right)_{j,k=1}^{N}$, $\omega = e^{2\pi i/N}$ N^{th} root of unity

$F\begin{pmatrix} f(0) \\ f(2\pi/N) \\ \vdots \\ f\left(\frac{(N-1)2\pi}{N}\right) \end{pmatrix}$ gives f approx as comb of $e^{ik\theta}$

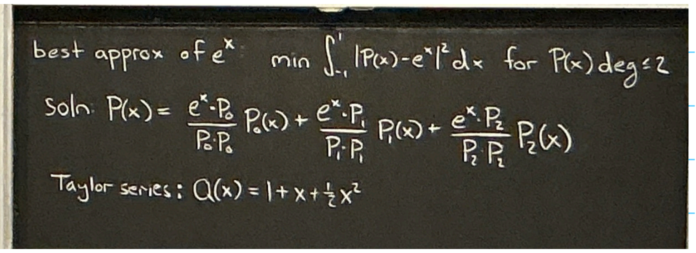
Legendre poly: $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

orthogonal on [-1,1]

$\int_{-1}^{1} f(x)g(x)\,dx$ is "$f \cdot g$"



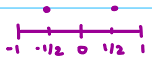
best approx of e^x: $\min \int_{-1}^{1} |P(x)-e^x|^2 dx$ for $P(x)$ deg ≤ 2

Soln: $P(x) = \frac{e^x \cdot P_0}{P_0 \cdot P_0} P_0(x) + \frac{e^x \cdot P_1}{P_1 \cdot P_1} P_1(x) + \frac{e^x \cdot P_2}{P_2 \cdot P_2} P_2(x)$

Taylor series: $Q(x) = 1 + x + \frac{1}{2}x^2$

$\int_{-1}^{1} f(x)\,dx \approx 2f(0)$ ← midpoint rule

EX: $\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$



note: $\int_{-1}^{1} a_0 + a_1 x\,dx = 2a_0$

*integrate linear func from -1 to 1 → multiply by 2 & get integral

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\int_{-1}^{1} p(x)\,dx = \left(a_0 + \frac{1}{3}a_2 x^2\right)\Big|_{-1}^{1} = 2a_0 + \frac{2}{3}a_2 x^2 = p(-1/\sqrt{3}) + p(1/\sqrt{3})$

↓ not on exam/pset

GAUSSIAN QUADRATURE: choose k points & k weights to integrate polys of deg < 2k exactly

$\int_{-1}^{1} p(x)\,dx = w_1 p(x_1) + \dots + w_k p(x_k)$ for p deg < 2k

CLAIM: k points are roots of $P_k(x)$ (kth Legendra poly) & weights soln to

$$\begin{pmatrix} P_0(x_1) & P_0(x_2) & \cdots & P_0(x_k) \\ P_1(x_1) & P_1(x_2 & \cdots & P_1(x_k) \\ & \vdots & & \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} P_0(x)\,dx \\ \\ \end{pmatrix}$$

on exam → MARKOV MATRICES + STOCHASTIC VECTORS

$\vec{p}$ is stochastic if 1) all entries ≥ 0
2) $\vec{p}^T \vec{1} = 1$

$A \in \mathbb{R}^{n \times n}$ is Markov if every column is stochastic

$\Rightarrow A^T \vec{1} = \vec{1} \Rightarrow$ 1 eval of A, $\vec{1}$ evec of A^T

note: A Markov, $\vec{p}$ Stochastic $\Rightarrow A\vec{p}$ stochastic

pf: $A\vec{p}$ has non-neg. entries and $(A\vec{p})^T \vec{1} = \vec{p}^T A^T \vec{1} = \vec{p}^T \vec{1} = 1$

also, A, B, Markov ⇒ AB Markov

EX: game → start w/ # m=0

if m is even, do nothing w/ prob. 1/3

add 1 w/ prob 2/3

if m is odd, double w/ prob 1/2

triple w/ prob 1/2

Q: what fraction of time is m odd?

$\vec{p}_k = \begin{pmatrix}\text{prob odd @ step k}\\ \text{prob even @ step k}\end{pmatrix}$, $\vec{p}_0 = \begin{pmatrix}0\\1\end{pmatrix}$, $A = \begin{array}{cc|l} 1/2 & & \to \text{odd} \\ 1/2 & 1/3 & \to \text{even} \\ \uparrow & \uparrow & \\ \text{if odd} & \text{if even} & \end{array}$

$\vec{p}_{k+1} = A\vec{p}_k \Rightarrow \vec{p}_k = A^k\vec{p}_0$

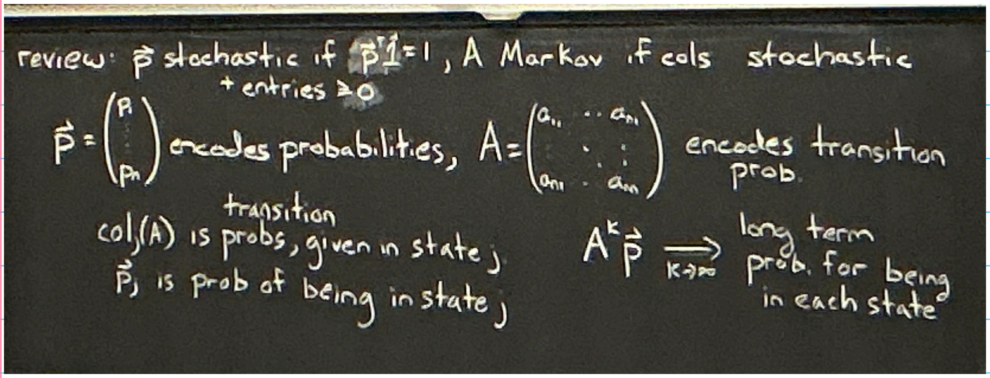
$\lambda_1 = 1$, $tr(A) = 1/2 + 1/3 = 5/6 = \lambda_1 + \lambda_3 \Rightarrow \overbrace{\lambda_2 = -1/6}$ → how?

$A - I = \begin{pmatrix}-1/2 & 2/3\\ 1/2 & -2/3\end{pmatrix} \to \vec{v}_1 = \begin{pmatrix}1\\3/4\end{pmatrix}$ | $A^k(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$

$= c_1\vec{v}_1 + c_2(-1/6)^k\vec{v}_2$

$\vec{p}_k \xrightarrow{k\to\infty} c\begin{pmatrix}1\\3/4\end{pmatrix} = \begin{pmatrix}4/7\\3/7\end{pmatrix}$

11/24 **MARKOV MATRICES + STOCHASTIC VECS**



review: $\vec{p}$ stochastic if $\vec{p}^T\vec{1} = 1$, A Markov if cols stochastic

\+ entries ≥ 0

$\vec{p} = \begin{pmatrix}p_1\\ \vdots\\ p_n\end{pmatrix}$ encodes probabilities, $A = \begin{pmatrix}a_{11} & \cdots & a_{n1}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ encodes transition prob.

$col_j(A)$ is transition probs, given in state j

$\vec{p}_j$ is prob of being in state j

$A^k\vec{p} \xrightarrow[k\to\infty]{}$ long term prob. for being in each state

Ex: <u>CAR RENTALS</u> = fraction of rental cars in Denver is 1/50. (49/50 not in Denver.)

each month, 80% of Denver cars stay in Denver, 5% outside cars come in.

what is asymptotic fraction of cars in Denver?

$\vec{p} = \begin{pmatrix}1/50\\ 49/50\end{pmatrix}\begin{matrix}\leftarrow \text{in Denver}\\ \leftarrow \text{not in Denver}\end{matrix}$ $A = \begin{array}{cc} 4/5 & 1/20 \\ 1/5 & 19/20 \\ \uparrow & \uparrow \\ \text{in Denver} & \text{not in Denver} \end{array}$

$A^k\vec{p}$ prob k month in future

$A = \begin{pmatrix}4/5 & 1/20\\ 1/5 & 19/20\end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix}4.5 - \lambda & 1/20\\ 1/5 & 19/20 - \lambda\end{vmatrix} = (4/5 - \lambda)(19/20 - \lambda) - 1/100 = 0 \to \lambda = 1, \overbrace{4/5 + 19/20}^{\text{trace}} - 1 = 3/4$

evec $\lambda = 1$: $A - I = \begin{pmatrix}-1/5 & -19/20\\ -4/5 & -1/20\end{pmatrix} \to \vec{v} = \begin{pmatrix}1\\4\end{pmatrix}$ $A^k\vec{p} \xrightarrow[k\to\infty]{} c\begin{pmatrix}1\\4\end{pmatrix}$ $c = \frac{1}{5}$ ← for stochastic, elts of vector must = 1, so $c = \frac{1}{5}$

Ex: $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ special Markov chain. everything moves to state 1.

note: A^k Markov ⇒ all eigenvals of A $|\lambda| \leq 1$

thm: if A is Markov w/ + entries, then there's a unique stationary vector $\vec{\pi}$ w/ $A\vec{\pi} = \vec{\pi}$,

$\vec{\pi}$ has + entries and $A^k \xrightarrow[k\to\infty]{} (\vec{\pi} \cdots \vec{\pi})$ $\begin{pmatrix}1 & 0 & 1/2\\ 0 & 1 & 1/2\\ 0 & 0 & 0\end{pmatrix}$

Ex: PAGERANK $n = \#$ web pages $\ell_j = \#$ urls page j links to

$A_{ij} = \begin{cases}1/\ell_j & \text{if page j links to page i}\\ 0 & \text{otherwise}\end{cases}$

A Markov not strictly positive

$G = cA + \frac{(1-c)}{n}\vec{1}\vec{1}^T \to \begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}$ $0 \leq c \leq 1$

↑ adds a bit of chance for url to go somewhere else

MINIMIZING MULTIVARIABLE FN

$F(\vec{x}): \mathbb{R}^n \to \mathbb{R}$

Taylor Series: $F(\vec{x}+\Delta\vec{x}) = F(\vec{x}) + \Delta\vec{x}\,\overrightarrow{\nabla F(\vec{x})} + \frac{1}{2}\overrightarrow{\Delta x}^T H(x)\overrightarrow{\Delta x} + O(\|\Delta\vec{x}\|^3)$

$\overrightarrow{\nabla F(\vec{x})} = \begin{pmatrix} \partial F/\partial x_1 \\ \vdots \\ \partial F/\partial x_n \end{pmatrix}$ gradient $\qquad [H(x)]_{ij} = \partial^2 F/\partial x_i \partial x_j$ Hessian

12/1 review: func. $F(\vec{x}): \mathbb{R}^n \to \mathbb{R}$ has quadratic Taylor series: $F(\vec{x}+\Delta\vec{x}) \approx F(\vec{x}) + \vec{\Delta} x^T\, \overrightarrow{\nabla F(x)} + \frac{\vec{\Delta} x^T H(\vec{x})\vec{\Delta} x}{2!}$

gradient $\overrightarrow{\nabla F(x)} = \begin{pmatrix} \partial F/\partial x_1 \\ \vdots \\ \partial F/\partial x_n \end{pmatrix}$, Hessian $H(\vec{x}) \in \mathbb{R}^{n\times n}$ with $H(\vec{x})_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$

Ex: $f(x,y) = x^2 + xy^2$ $\quad \nabla F = \begin{pmatrix} 2x+y^2 \\ 2xy \end{pmatrix}$, $H(\vec{x}) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$

taylor series about 0 $f(x,y) \approx 0 + \underset{\text{eval. @ 0}}{(x\ y)\begin{pmatrix} 0 \\ 0 \end{pmatrix}} + \frac{1}{2}(x\ y)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2$

GOAL: minimize multivar. fn $\quad$ min $F(\vec{x}): \mathbb{R}^n \to \mathbb{R}$

if F is convex ⟶ minimum occurs @ point $\vec{x}_*$ where $\nabla F(\vec{x}_*) = \vec{0}$

also perform Taylor series through gradient $\overrightarrow{\nabla F(x)}$: $\overrightarrow{\nabla F(\vec{x}+\Delta\vec{x})} = \overrightarrow{\nabla F(\vec{x})} + H(\vec{x})\Delta\vec{x}$

to find $\nabla F(\vec{x}+\Delta\vec{x}) = \vec{0}$, guess

$\vec{0} = \overrightarrow{\nabla F(x)} + H(\vec{x})\Delta\vec{x}$

$H(\vec{x})\Delta\vec{x} = \overrightarrow{\nabla F(\vec{x})} \Rightarrow \Delta\vec{x} = H(\vec{x})^{-1}\overrightarrow{\nabla F(x)} \Rightarrow$ "$\Delta\vec{x} = H(\vec{x})^{-1}\overrightarrow{\nabla F(\vec{x})}$"

NEWTONS METHOD: $\vec{x}_{k+1} = \vec{x}_k - H(\vec{x}_k)^{-1}\,\overrightarrow{\nabla F(\vec{x}_k)}$

PRO: very fast

CON: $H(\vec{x})$ must be invertible, convexity

Ex: min $\frac{1}{3}x^3 - 4x$ on $[0, \infty)$

$\nabla F(x) = x^2 - 4 \qquad H(x) = 2x$

Newton: $x_{k+1} = x_k - (2x_k)^{-1}(x_k^2 - 4)$

$= x_k - \frac{x_k^2 - 4}{2x_k}$

cheaper idea: make $F(\vec{x})$ smaller by moving in direction opposite → $-\overrightarrow{\nabla F(\vec{x})}$

$x_0 = 4$, $x_1 = \frac{4}{2} + \frac{2}{4} = \frac{5}{2}$, $x_2 = \frac{5/2}{2} + \frac{2}{5/2} = \frac{41}{20}$, $x_3 \approx 2.0006$, $x_4 \approx 2.000000009$

GRADIENT DESCENT: $\vec{x}_{k+1} = \vec{x}_k - \vec{s}_k\,\overrightarrow{\nabla F(\vec{x}_k)}$ for step size $s_k \in \mathbb{R}$

Ex: $F(x,y) = x^2 + y^2$, guess $x_0 = y_0 = 1$

$\vec{\nabla} F = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, $H(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$

GD: $\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - 2s\begin{pmatrix} x_k \\ y_k \end{pmatrix} = (1-2s)\begin{pmatrix} x_k \\ y_k \end{pmatrix}$

if $s = 1/2$: $x_1 = 0$, $y_1 = 0$, $s = 1$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$… oscillates ← step size too big - never converges

$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = (1-2s)^k\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |s|<1 \Rightarrow$ kth step $\left\|\begin{pmatrix} x_k \\ y_k \end{pmatrix}\right\| = \sqrt{2}(1-2s)^k$

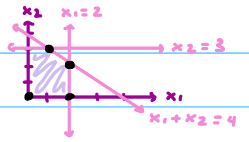
NEWTONS: $x_{k+1} = \vec{x}_k - H(\vec{x}_k)^{-1}\,\overrightarrow{\nabla F(\vec{x}_k)}$

$= \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2x_k \\ 2y_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

LINEAR PROGRAMMING

$\min \vec{c}^T\vec{x}$ subject to $\overset{\text{entry wise}}{A\vec{x} \le \vec{b},\ \vec{x} \ge 0}$

Ex: $\min -3x_1 - 2x_2$ s.t. $\begin{cases} x_1 + x_2 \le 4 \\ x_1 \le 2 \\ x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$



start @ (0, 0), move along edges

(0, 0) → (2, 0) → (2, 2) → (1, 3)

$-3(0)-2(0)=0$ $-3(2)-2(0)=-6$ $\underbrace{-3(2)-2(2)=-10}_{\text{OPTIMUM!}}$ $-3(1)-2(3)=-9$

SIMPLEX ALGORITHM: find corner, move along edge where $\vec{c}^T\vec{x}$ decreases, repeat.

→ used when solving linear system

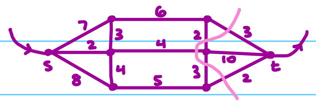
12/3 **LINEAR PROGRAMMING:**

$\min \vec{c}^T\vec{x}$ s.t. $A\vec{x} \le \vec{b}$, $\vec{x} \ge 0$

dual: $\max \vec{y}^T\vec{b}$ s.t. $A^T\vec{y} \ge \vec{c}$, $\vec{y} \ge 0$

$\max \vec{y}^T\vec{b} \ge \vec{y}A\vec{x} = (A^T\vec{y})^T\vec{x} \ge \vec{c}^T\vec{x}$ (in LP class, see (1), (2) have same soln)

Ex: mincut, max flow



max flow on edges : flow @ most capacity @ each edge, @ each vertex, flow in = flow out

MIN CUT ON EDGES: minimize sum of weights of cut edges that seperates s from t

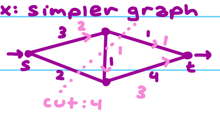
by linear prog, these should be equal

flow: 4

14

cut: 15

Ex: simpler graph



$\max f_{sa} + f_{sb}$ s.t. $-3 \le f_{sa} \le 3$, $-4 \le f_{bt} \le 4$

$-2 \le f_{sb} \le 2$

$-1 \le f_{ab} \le 1$

$-1 \le f_{at} \le 1$

$f_{sa} = f_{ab} + f_{at}$

$f_{sb} + f_{ab} = f_{bt}$

Ex: two player games

player R & C, payoff matrix A

player R chooses a row i, player C chooses col j, result A_{ij}

R wants min result, C wants to maximize result

$$\begin{array}{c|ccc} & \text{rock} & \text{paper} & \text{scissor} \\ \hline \text{rock} & 0 & 10^6 & -10^6 \\ \text{paper} & -10^6 & 0 & 10^6 \\ \text{scissor} & 10^6 & -10^6 & 0 \end{array}$$

Ex: $\begin{pmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{pmatrix}$ $x_1+x_2=1$ $x_1, x_2=0$

$y_1+y_2+y_3=1,$ $y_1, y_2, y_3=0$

R chooses x s.t. $\begin{cases} 1\cdot x_1+3\cdot x_2 = 4\cdot x_1+2\cdot x_2 \\ x_1+x_2=1 \end{cases} \Rightarrow \begin{matrix} x_1 = 1/4 \\ x_2 = 3/4 \end{matrix}$ payoff $= 1(1/4)+3(3/4) = 5/2$

C chooses y s.t. $\begin{cases} 1y_1+4y_2 = 3y_1+2y_1 \\ y_1+y_2 = 1 \end{cases} \Rightarrow \begin{matrix} y_1 = 1/2 \\ y_2 = 1/2 \end{matrix}$ payoff $= 1(1/2)+4(1/2) = 5/2$

for bigger $A \in \mathbb{R}^{m \times n}$, use LP:

R's LP: min v s.t. $\vec{x} \geq 0$, $\vec{x}^T \vec{\mathbb{1}} = 1$, $\vec{x}^T A \geq v \vec{\mathbb{1}}^T$

C's LP: max v s.t. $\vec{y} \geq 0$, $\vec{y}^T \vec{\mathbb{1}} = 1$, $v\vec{\mathbb{1}} \leq A\vec{y}$

12/8 **BIG PIC OF LINEAR ALG**

objects: vectors + linear transformations

$\mathbb{R}^n, \mathbb{C}^n$ matrices, poly, etc. — any set closed under $a\vec{x}+b\vec{y}$

fn satisfies $T(a\vec{x}+b\vec{y}) = aT(\vec{x})+bT(\vec{y})$

BASIS: coord. sys. for vectors, allows lin. trans to be represented by matrices

3 BIG IDEAS:

1) pre-image of lin. trans. $(A\vec{x}=\vec{b})$

2) best approx.

3) finding a good basis

1: **PRE-IMAGE OF LIN. TRANS:**

want to solve $A\vec{x}=\vec{b}$

if $\mathbb{R}^{n \times n}$ invertible $\rightarrow$ $\underset{\text{perm}}{P}A=\underset{\substack{\text{lower} \\ \Delta}}{L}\underset{\substack{\text{upper} \\ \Delta}}{U}$ fact. $A \rightarrow U = \overset{\text{elim matrix}}{(E_{n,n-1} \dots E_{31}, E_{21})} A$ (if no zero pivots)

Ex: 3x3 matrix that subtracts $2 \cdot \text{row}_1$ from row 3:

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $A = \overbrace{(E_{m,n-1}, \dots, E_{2,1})^{-1}}^{L} U$

if not square + invertible, more complicated

useful subspaces: $N(A) = \{\vec{x} \mid A\vec{x}=0\}$, $C(A) = \{A\vec{x}\}$

$\text{rank}(A) = \dim(C(A))$

FTLA: $N(A)^{\perp} = C(A^T)$, $C(A) = N(A^T)$

BASIS: lin. ind. vectors that span space

full soln to $A\vec{x}=\vec{b}$: $\vec{x}_p + c_1\vec{s}_1 + \dots + c_k\vec{s}_k$, where

$A\vec{x}_p = \vec{b}$ & $N(A) = \text{span}\{\vec{s}_1, \dots, \vec{s}_k\}$

RREF: tool for finding $\vec{x}_p$ & basis for N(A)

★ Ex: $(A|\vec{b}) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \overset{EA \qquad E\vec{b}}{\left(\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}$ $\vec{x}_p = \begin{pmatrix} 7 \\ 0 \\ -3 \\ 0 \end{pmatrix}$ $\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{s}_2 = \begin{pmatrix} 4 \\ 0 \\ -4 \\ 1 \end{pmatrix}$

2: BEST APPROX

best basis: orthonormal $\vec{q}_1, \dots, \vec{q}_k$ $\vec{q}_i \cdot \vec{q}_j = 0$ $i \neq j$ $\|\vec{q}_i\| = 1$

orthog proj. onto span$\{\vec{q}_1, \dots, \vec{q}_k\}$ is QQ^T, where $Q = (\vec{q}_1, \dots, \vec{q}_k)$

if Q is n×n & real → orthogonal

if Q is complex → unitary

GRAM SCHMIDT: builds orthog. basis

$\{\overset{\text{basis}}{\vec{x}_1, \dots, \vec{x}_k}\} \longrightarrow \{\overset{\text{orthog.}}{\vec{v}_1, \dots, \vec{v}_k}\}$

$\vec{v}_1 = \vec{x}_1$, $\vec{v}_2 = \vec{x}_2 - \frac{\vec{v}_1 \vec{v}_1^T}{\vec{v}_1^T \vec{v}_1} \vec{x}_2$, $\vec{v}_3 = \vec{x}_3 - \frac{\vec{v}_1 \vec{v}_1^T}{\vec{v}_1^T \vec{v}_1} \vec{x}_3 - \frac{\vec{v}_2 \vec{v}_2^T}{\vec{v}_2^T \vec{v}_2} \vec{x}_3$

LEAST SQUARES: if $A\vec{x} = b$ has no solns and $N(A) = \emptyset$, soln that minimizes $\|A\vec{x} - \vec{b}\|^2$ is

unique soln to $A^T A \vec{x} = A^T \vec{b}$ and $A\vec{x}$ is proj of $\vec{b}$ onto C(A)

DET: check vol of parallelpipeds + checking invertibility

→ if matrix is triangular: det = product of diagonal elts

3: FINDING A GOOD BASIS

★ if $A\vec{x} = \lambda\vec{x}$ for $\vec{x} \neq 0$, $\begin{matrix} \vec{x} = \text{evec} \\ \lambda = \text{eval} \end{matrix}$ of A

if A has basis of evecs $\vec{x}_1, \dots, \vec{x}_n$ w/ evals $\lambda_1, \dots, \lambda_n$ $\Rightarrow AX = X\Lambda$

$X = (\vec{x}_1 \cdots \vec{x}_n)$, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow A = X\Lambda X^{-1}$, $\Lambda = X^{-1}AX$

makes it easy to take powers: $A^k = X\Lambda^k X^{-1}$ ← Markov chains

$e^{At} = Xe^{\Lambda t}X^{-1}$ ← diff. eqs

CHANGE OF BASIS:

new input $V = (\vec{v}_1 \cdots \vec{v}_n)$

new output $W = (\vec{w}_1 \cdots \vec{w}_m)$

new matrix is $W^{-1}AV$, $V = W = \underset{\text{e.vecs}}{X} \rightarrow X^{-1}(X\Lambda X^{-1})X = \Lambda$

★ **when can we find a good basis?**

- distinct evals → $A = X\Lambda X^{-1}$
- symm. mat $\xrightarrow{\text{spectral thm}}$ $A = \overset{\text{orthog.}}{Q}\Lambda Q^T$
- any m×n matrix $\xrightarrow{\text{SVD}}$ $A = \overset{\text{orthog,}}{U}\Sigma V^T$

  useful for rank approx.
  Eckart-Young, PCA direction $\vec{u}_1$

how to compute:

$A = U\Sigma V^T$ diagonalize $A^TA = V(\Sigma^T\Sigma)V^T$, $U\Sigma = AV$

Ex: $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 2, 4$

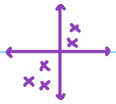
$A^TA = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ $\quad A^TA - 4I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \ \sigma_1 = 2$

$A^TA - 2I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \ \sigma = \sqrt{2}$

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2/\sqrt{2} \\ 2/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$ ...

12/10 PCA: $A = (\vec{x}_1 \dots \vec{x}_5)$ $\quad A = \overset{\vec{u}_1}{U}\Sigma V^T$



$AA^T \in \mathbb{R}^{2\times 2}$ $\quad AA^T = U\Sigma V^T(U\Sigma V^T)^T$

$A\vec{\mathbb{1}} = \vec{0}$ $\quad = U\Sigma V^T V\Sigma U^T = U(\Sigma\Sigma^T)U^T$

how many solns does $A\vec{x} = \vec{b}$ have? 0, 1, ∞ many

$\vec{b} \notin C(A) \Rightarrow 0$

$\vec{b} \in C(A)$ — $N(A) = \{\vec{0}\} \rightarrow 1$

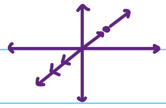
— $N(A) \neq \{\vec{0}\} \rightarrow \infty$

least sq: $\nexists \vec{x}$ s.t. $A\vec{x} = \vec{b} \iff \vec{b} \notin C(A)$

$e^{\lambda_1 t}, \vec{v}_1$ $\quad (\vec{v}_1 \ \vec{v}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{x}$

$e^{\lambda_2 t}, \vec{v}_2$

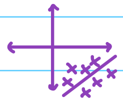
$\vec{x}$



find $\vec{x}$ min $\|A\vec{x} - \vec{b}\|^2$ is exactly given by $A\vec{x} = \text{proj}_{C(A)}\vec{b}$

$A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$

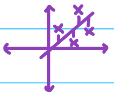
$(\ )^{(\ )} = (\ )$ $\quad N(A) = \{\vec{0}\}$ $\quad AA\vec{x} = A^T\vec{b}$ $\quad \|a\vec{x}\| = |a|\,\|\vec{x}\|$



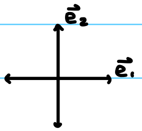
$A = U\Sigma V^T$

$cA = U(c\Sigma)V^T, \ c > 0$ $\quad \frac{1}{n-1}AA^T$

$(cA)(cA^T) = U(c^2\Sigma\Sigma^T)U^T$



$\vec{x}, \vec{y} \in V$ is $a\vec{x} + b\vec{y} \in V$



change of basis $\{\vec{e}_1, \vec{e}_2\} \rightarrow \{\vec{v}_1, \vec{v}_2\}$ $\quad$ Q: $\vec{x} = \beta_1\vec{v}_1 + \beta\vec{v}_2$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$ $\quad \vec{x} = [\vec{v}_1 \ \vec{v}_2]\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$\beta = V^{-1}\vec{x}$

$A \in \mathbb{R}^{m\times n}$

$\mathbb{R}^m \ \{e_1 \cdots e_m\} \ A \quad \mathbb{R}^n \ \{\vec{e}_1, \dots \vec{e}_n \quad M \ \{\vec{v}_1 \dots \vec{v}_n\}$

think about same transform w/

$M\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} 4^{Kth} = \vec{v}_K \Rightarrow M = V$

input: $V = (\vec{v}_1 \cdots \vec{v}_n)$

output: $W = (\vec{w}_1 \cdots \vec{w}_m)$

$A = X\Lambda X^{-1} \rightarrow X^{-1}AX = X^{-1}(X\Lambda X^{-1})X = \Lambda$

$X = (\vec{x}_1 \cdots \vec{x}_n)$ $\quad A = U\Sigma V^T$ $\quad U^TAV = \overbrace{U^T(U}^{I}\Sigma\overbrace{V^T)V}^{I} = \Sigma$

"V" = V, "W" = U

$(V = W = X)$